Fakultät für Mathematik
Universität Regensburg

Seminar
WS 2018/19

K-THEORY AND TOPOLOGICAL CYCLIC HOMOLOGY OF HENSELIAN PAIRS

Tuesday, 14-16, M 311 (tbc!)

Introduction

The goal of this seminar is to work through the recent preprint by Clausen–Mathew–Morrow on *K-theory and topological cyclic homology of henselian pairs* [CMM]. For any ring R, the cyclotomic trace map $\mathrm{K}(R) \to \mathrm{TC}(R)$ is a map from algebraic K-theory to topological cyclic homology and is known to be "locally constant". This just means that in many situations, one can get information about relative algebraic K-theory by means of topological cyclic homology. A concrete example of such a phenomenon is the following: Let K^{inv} be the fibre of the cyclotomic trace and let $R \to S$ be a surjection of rings with nilpotent kernel. The celebrated theorem of Dundas–Goodwillie–McCarthy [DGM, Theorem 7.0.0.2] says that in this situation, the canonical map

$$K^{\mathrm{inv}}(R) \to K^{\mathrm{inv}}(S)$$

is an equivalence. A precursor of this theorem deals with the statement that this map is an equivalence modulo n for every $n \geq 1$, and is due to McCarthy [McC].

Related, but slightly different to this situation, there is Gabber's rigidity theorem: It says that given a henselian pair (A, I) and assuming that n is invertible in A, then the canonical map

$$K(A)/n \to K(A/I)/n$$

is an equivalence.

The insight of [CMM] is to see that the (not obviously related) theorems of McCarthy and Gabber are in fact special cases of the following general theorem, the main result of [CMM]:

Main Theorem. *Let (A, I) be a henselian pair. Then for any $n \geq 1$, the map $K^{\mathrm{inv}}(R)/n \to K^{\mathrm{inv}}(R/I)/n$ is an equivalence.*

Indeed, Gabber's result follows from this since TC of a $\mathbb{S}[\frac{1}{n}]$-algebra, is itself an $\mathbb{S}[\frac{1}{n}]$-algebra and thus vanishes modulo n. Setting the stage (talks 1–6) and proving this theorem (talks 7–11) will occupy most of the seminar. However, also many interesting consequences of this theorem are established in [CMM], and we intend to cover some of those as well (talks 12–15):

- (Continuity of p-adic K-theory) For an I-adically complete, noetherian ring R such that R/p is F-finite, the canonical map $K(R) \to \lim K(R/I^n)$ is a p-adic equivalence.
- (Étale K-theory is TC at points of characteristic p) For a strictly henselian local ring of residue characteristic p, the map cyclotomic trace is an equivalence; in other words in this situation TC is identified with p-adic étale K-theory.
- (Asymptotic comparison of K and TC) For a ring R, henselian along (p) such that R/p has finite Krull dimension, the cyclotomic trace induces an isomorphism between

mod p K-groups and mod p TC-groups in sufficiently high degrees, in light of the above this can be viewed as a kind of Quillen–Lichtenbaum theorem.

The proof of the Main Theorem relies on various inputs:

- The classical Dundas–Goodwillie–McCarthy theorem that K^{inv} is invariant under nilpotent thickenings, [DGM, Theorem 7.0.0.2],
- a theorem of Geisser–Levine describing the mod p K-theory of ind-smooth local $\mathbb{F}_p$-algebras in terms the inverse Cartier operator on differential forms [GL], and
- a theorem of Geisser–Hesselholt describing TC and the cyclotomic trace in this situation [GH].

We suggest to use these theorems as black boxes. In addition to the above, one uses various finiteness properties of TC that are established in section 2 of [CMM] and which we believe to be of independent interest. We suggest to include those in the program. The seminar will be divided roughly into three parts: In the first part we recall the notion of cyclotomic spectra, construct the cyclotomic trace map and deal with the finiteness properties of TC. The second part consists of the proof of the main theorem, and the third part consists of the mentioned applications. A detailed plan for talks is given below.

Prerequisits and Organization

Homotopy theory, algebraic geometry, algebraic K-theory as well as the language of ∞-categorieswill be used quite freely in the seminar. Some background with cyclotomic spectra and topological cyclic homology is helpful, but we will spend some time on recollections about TC.

If you are interested in giving a talk, please send an email to “markus.land@ur.de” and indicate which of the talks you would be interested in, so that we can distribute the talks soon.

Cyclotomic spectra and the cyclotomic trace

Talk 1: Cyclotomic spectra and topological cyclic homology.
Recall/introduce the category CycSp of cyclotomic spectra and define the topological cyclic homology $\mathrm{TC}(X)$ of a cyclotomic spectrum X. Briefly explain how $\mathrm{THH}(R)$ is a cyclotomic spectrum if R is an $\mathbb{E}_1$-ring (you can also stick to the $\mathbb{E}_\infty$-case if you wish – this makes the definition easier). Explain how to define $\mathrm{TR}^n(X)$ for a cyclotomic spectrum, and how to obtain TC from the TR^n’s. References are [CMM, §2.1], [NS, II.1].

Talk 2: Topological Hochschild Homology of stable ∞-categories.
Present the construction of topological Hochschild homology (as a cyclotomic spectrum) for stable ∞-categories. Show some of the important properties of this functor $\mathrm{THH}\colon \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{CycSp}$ such as Morita invariance and that it is localizing. The reference here is [AG, pp. 53-61] and possibly [Nik].

Talk 3: The cyclotomic trace.
Construct the cyclotomic trace $K(\mathcal{C}) \to \mathrm{TC}(\mathcal{C})$ for stable ∞-categories. If you wish you can discuss the universal property of K-theory on non-commutative motives and indicate why all constructions of cyclotomic traces are equivalent. Finish the talk by stating the theorem of Dundas–Goodwillie–McCarthy [DGM, Theorem 7.0.0.2] (in its correct form). References are [AG, pp. 70-74] and [BGT, §10.3].

The finiteness property of TC

Talk 4: Cocontinuity of TC/p.
Prove that the functor $\mathrm{TC}/p\colon \mathrm{CycSp}_{\geq 0} \to \mathrm{Sp}$ commutes with all colimits. This is [CMM, Theorem 2.7] and discussed in [CMM, §2.2].

Talk 5: Further finiteness of TC/p.
Deduce from the last talk that the functor $R \mapsto \mathrm{TC}(R)/p$ from *connective* $\mathbb{E}_1$-rings to Sp commutes with sifted colimits and explain [CMM, Example 2.18]. Then go on to discuss the further properties of TC/p given in [CMM, §2.3] in particular [CMM, Proposition 2.19] which will be needed later.

Talk 6: TC/p via the de Rham–Witt complex.
Relate $\mathrm{TC}(R)$ for smooth $\mathbb{F}_p$-algebras to the de Rham–Witt complex and deduce that the functor $\mathrm{TC}/p\colon \mathrm{Alg}_{\mathbb{F}_p} \to \mathrm{Sp}$ commutes with filtered colimits. This is [CMM, §2.4]. For Hesselholt's HKR-result it might be worthwhile to consult the recent preprint of Antieau–Nikolaus, [AN].

Proof of the main result

Talk 7: Henselian pairs.
Define henselian pairs, relate them to henselian nonunital rings and discuss henselisation, following [CMM, §3]. This is a lot of material to cover and many aspects are very formal, so you will need to decide which of the aspects you want to explain in (some amount of) detail and which to only state.

Talk 8: Pseudo-coherent functors.
Introduce pseudocoherent functors and show that the functors K and TC/p are projectively pseudocoherent. Again, this is a lot of material, so choose wisely what to explain in (some amount of) detail and where to be brief. This is [CMM, §4.1, §4.3].

Talk 9: Axiomatic rigidity.
Show the axiomatic rigidity theorem for functors with values in spectra and rigidity for functors with values in abelian groups as discussed in [CMM, §4.2].

Talk 10: Gabber rigidity.
Prove the part of Gabber rigidity that we will need in the seminar, i.e. prove that for a prime field k, of characteristic prime to p, we have that $K(k\{x_1, \dots, x_n\})/p \to K(k)/p$ is an equivalence. Here, $k\{x_1, \dots, x_n\}$ denotes the free henselian k-algebra on n elements, i.e. the henselization of $k[x_1, \dots, x_n]$. References are [Gab, proof of Theorem 2, page 66] and [GT, Corollary 4.2 & Note 4.3]

Talk 11: Proof of the Main Theorem.
Prove that for any henselian pair (R, I) the map $K^{\mathrm{inv}}(R) \to K^{\mathrm{inv}}(R/I)$ becomes an equivalence modulo any prime number and deduce the Main Theorem. The proof relies on Gabber's rigidity theorem and results of Geisser-Levine and Geisser-Hesselholt as explained in the introduction; the latter ones have to be taken for granted. Also you will need the fact that K^{inv} satisfies excision, but you can explain how one deduces this from [DGM, Theorem 7.0.0.2]

as in [LT, Theorem B]. Convince yourself that this is the only point where the Dundas–Goodwillie–McCarthy theorem enters the proof of the main theorem.
[CMM, §4.4, §4.5].

Continuity and pro statements for algebraic K-theory

Talk 12: Continuity of K-theory.
Prove that K-theory satisfies continuity modulo p for complete noetherian rings R such that R/p is F-finite, [CMM, Theorem 5.5]. Explain the failure of continuity without the assumption of F-finiteness [CMM, Theorem 5.7]. All of this is covered in [CMM, §5.1].

Talk 13: p-adic continuity and K-theory.
Explain the results of [CMM, §5.2], most notably [CMM, Theorems 5.10, 5.22 and 5.23]. The latter says that K-theory is p-adically continuous at R if the pair $(R,(p))$ is henselian and R has bounded p-power torsion.

Comparison of K and TC

Talk 14: Étale K-theory is TC.
Following [CMM, §6.1], Prove that the map $\mathrm{K}(R) \to \mathrm{TC}(R)$ is a p-adic equivalence if R is a strictly henselian local ring of residue characteristic p. Deduce the global version of this result for proper R-schemes. If you have time, explain why TC satisfies étale descent.

Talk 15: Asymptotic comparison of K and TC.
Show that under suitable conditions the cyclotomic trace is an equivalence modulo p in large enough degrees, [CMM, Theorem 6.5]. Discuss the examples given in [CMM, §6.3]
[CMM, §6.2, §6.3]

References

[AN] B. Antieau, T. Nikolaus: *Cartier modules and cyclotomic spectra* preprint.

[AG] Report to *Arbeitsgemeinschaft: Topological Cyclic Homology* at Mathematisches Forschungsinstitut Oberwolfach, 1-7 April 2018. DOI: 10.4171/OWR/2018/15

[BGT] A. J. Blumberg, D. Gepner, and G. Tabuada: *A universal characterization of higher algebraic K-theory.* Geom. Topol. 17 (2013), no. 2, 733-838.

[CMM] D. Clausen, A. Mathew, and M. Morrow: *K-theory and topological cyclic homology of henselian pairs.* `arXiv:1803.10897`.

[DGM] B. I. Dundas, T. G. Goodwillie, and R. McCarthy: *The local structure of algebraic K-theory.* Algebra and Applications, vol. 18, Springer, 2013.

[Gab] O. Gabber: *K-theory of Henselian local rings and Henselian Pairs.* In *Algebraic K-theory, commutative algebra, and algebraic geometry*, volume 126 of *Cont. Math.* pages 59-70, Amer. Math. Soc., Providence, RI, 1992.

[GH] T. Geisser, L. Hesselholt: *Topological cyclic homology of schemes* In *Algebraic K-theory (Seattle, WA, 1997)* volume 67 of *Proc. Sympos. Pure Math.*, Amer. Math. Soc., Providence, RI, 1999.

[GL] T. Geisser, M. Levine: *The K-theory of fields in characteristic p* Invent. Math., 139 (3), 2000.

[GT] H. A. Gillet and R. W. Thomason: *The K-theory of strict Hensel local rings and a theorem of Suslin.* Journal of Pure and Applied Algebra, 34 (2-3): 241–254, 1984

[LT] M. Land, G. Tamme: *On the K-theory of pullbacks* ArXiv:1808.05559

[McC] R. McCarthy: *Relative algebraic K-theory and topological cyclic homology* Acta Math., 179, (1997)

[Nik] T. Nikolaus: *Topological Hochschild homology and cyclic K-theory.* In preparation.

[NS] T. Nikolaus and P. Scholze: *On topological cyclic homology.* `arXiv:1707.01799`.