Fakultät für Mathematik
Universität Regensburg

$K$-theory Seminar
SS 2019

# GALOIS DESCENT IN TELESCOPICALLY LOCALIZED ALGEBRAIC $K$-THEORY

Tuesday, 14-16, M 311

## Introduction

The goal of this terms $K$-theory seminar is to go through the paper "Descent in algebraic $K$-theory and a conjecture of Ausoni–Rognes" of Clausen–Mathew–Naumann–Noel, [CMNN]. The setup is as follows: Starting with a $G$-Galois ring extension $A \to B$ in the sense of Rognes, there is a canonical map $K(A) \to K(B)^{hG}$, and Galois descent in algebraic $K$-theory asks how close this map is to being an equivalence. As stated, $K$-theory does not satisfy Galois descent in general. However, the situation becomes much better when one rationalizes, i.e. considers the map $K(A) \otimes \mathbb{Q} \to K(B)^{hG} \otimes \mathbb{Q}$ instead. Noting that rationalization is the localization at $T(0)$, the zeroth telescopic localization, one can ask how the situation behaves at higher chromatic heights, i.e. when considering the map

$$K(A) \otimes T(n) \to K(B)^{hG} \otimes T(n)$$

for $n \geq 1$ where $T(n) = X[v^{-1}]$ with $(X, v)$ a type $n$-complex with a $v_n$-self map $v$, i.e. a self map $v$ inducing an isomorphism in Morava $K$-theory $K(n)$. The case $n = 1$ is contained in a stronger result of Thomason: He considered noetherian schemes $X$ of finite Krull dimension, defined over $\mathbb{Z}[1/\ell, \mu_\ell]$ and satisfying a technical assumption which implies that all residue fields have uniformly bounded mod $\ell$ Galois cohomological dimension. In this case he observed that $K(-)/\ell$ has a Bott element $\beta \in \pi_2(K(-)/\ell)$ and he showed that the association

$$X \mapsto K(X)/\ell[\beta^{-1}]$$

satisfies étale hyperdescent (which implies ordinary Galois descent). Now there is an equivalence $K(X)/\ell[\beta^{-1}] \simeq K(X) \otimes (\mathbb{S}/\ell)[v_1^{-1}]$, where $v_1$ is a $v_1$-self map of $\mathbb{S}/\ell$. Since $T(1) = \mathbb{S}/\ell[v_1^{-1}]$, we find that the association $X \mapsto L_{T(1)}K(X)$ in particular satisfies étale descent.

For ordinary rings and schemes this picture continues trivially when enlarging the chromatic height: It is a result of Mitchell's that $L_{T(n)}K(X)$ vanishes for all $n \geq 2$, [Mit]. For ring spectra, however, the situation is more delicate and higher chromatic phenomena are present in algebraic $K$-theory. In this direction, there is the following conjecture of Ausoni–Rognes:

**Conjecture 1.** *Let $A \to B$ be a $K(n)$-local $G$-Galois extension of $\mathbb{E}_\infty$-ring spectra. Then the map*

$$L_{T(n+1)}(K(A)) \to L_{T(n+1)}(K(B)^{hG})$$

*is an equivalence. In other words, in this situation, $T(n)$-localized $K$-theory satisfies Galois descent.*

The main results in the paper [CMNN] are then as follows:

**Theorem 1.** *Let $A$ be an $\mathbb{E}_\infty$-ring spectrum and let $B$ be an $\mathbb{E}_\infty$-$A$-algebra such that $\pi_*(B)$ is finite and faithfully flat over $\pi_*(A)$. Then*

$$L_{T(n)}K(A) \simeq \mathrm{Tot}(L_{T(n)}K(B^{\otimes_A^\bullet}))$$

*i.e. $T(n)$-localized $K$-theory satisfies descent along $A \to B$.*

**Theorem 2.** *Let $A \to B$ be a map of $\mathbb{E}_\infty$-ring spectra. Assume that $B$ is a perfect $A$-module and that the transfer map $K_0(B) \to K_0(A)$ is rationally surjective. Then for $T$ any of the spectra $T(n), K(n), E_n$, the map*

$$L_T K(A) \to \mathrm{Tot}(L_T K(B^{\otimes_A^\bullet}))$$

*is an equivalence. Moreover, the Tot-spectral sequence for the right hand side collapses at a finite page with horizontal vanishing line.*

For the case of a $K(n)$-local $G$-Galois extension $A \to B$ as in the Ausoni–Rognes conjecture, the Tot-spectral sequence identifies with the homotopy fixed point spectral sequence. Hence, if the Galois extension $A \to B$ induces a surjection on rationalized $K_0$, then the Ausoni–Rognes conjecture holds true.

## Prerequisits and Organization

The language of $\infty$-categories will be used freely in the seminar. Previous knowledge about algebraic $K$-theory is helpful (though much of the material works equally well for any localizing invariant), the final two talks will deal with Galois extensions of $TMF$, so some familiarity with its properties will be helpful as well. Some chromatic homotopy theory is needed, but we will try to review all necessary inputs (the statements, of course, not the proofs).

If you are interested in giving a talk, please send an email to "markus.land@ur.de" or "georg.tamme@ur.de" and indicate which of the talks you would be interested in, so that we can distribute the talks soon.

## Detailed Program for Talks

**Talk 1: 30.04.2019 – Background from chromatic homotopy theory.**
Give a recollection of the results of Devinatz–Hopkins–Smith, and Hopkins–Smith about $v_n$-periodic maps on complexes of type $n$. In particular the nilpotence theorem is important for the later parts of the seminar. You might also want to explain the telescope conjecture. Focus on giving a nice presentation of the results rather than explaining proofs. These are for another seminar. References are [DHS, HS, Lur].

**Talk 2: 07.05.2019 – $\varepsilon$-Nilpotence I.**
Use the nilpotence technology of the previous talk and explain section 2.1 of [CMNN] including Corollary 2.9.

**Talk 3: 14.05.2019 – $\varepsilon$-Nilpotence II.**
Start with [CMNN, Example 2.11] and finish with the proof of Proposition 2.20.

**Talk 4: 21.05.2019 – $A$-nilpotence.**
Explain [CMNN, Section 2.3]. In the paper this is very short, but many points are outsourced to [MNN17], so explain some more background on $A$-nilpotence from that paper.

**Talk 5: 28.05.2019 – $A$-linear non-commutative motives.**
Explain [CMNN, Section 3]. This is largely formal properties of the category of $A$-linear non-commutative motives (including a definition).

**Talk 6: 04.06.2019 – The May conjecture.**
Explain the statement and resolution of May's conjecture on nilpotence of elements $x \in \pi_*(R)$ for an $\mathbb{E}_\infty$-ring $R$ following [MNN15]. You don't have to explain the notion of $H_\infty$-ring spectra: Just replace $H_\infty$ with $\mathbb{E}_\infty$-throughout.

**Talk 7: 18.06.2019 – Abstract descent results.**
Explain [CMNN, Section 4]. In the proof of Theorem 4.5 you will need the notion of acyclizations and localizations. This is explained in [MNN17, Sections 2 and 3], Prop. 2.21 will be needed in talk 10. You may use the Hopkins–Mahowald result on free $\mathbb{E}_2$-rings of characteristic $p$, but if you want to and have time for it, you can explain the constructions. References for this are [MNN15, Theorem 4.18], [ACB, Theorem 5.1], [KN, Appendix A].

**Talk 8: 25.06.2019 – Examples: The algebraic case.**
Explain [CMNN, Section 5.1] including [CMNN, Theorem 5.1].

**Talk 9: 02.07.2019 – Examples: Rognes' Galois extensions.**
Explain [CMNN, Section 5.2].

**Talk 10: 09.07.2019 – Examples: Further Galois extensions I.**
Explain [CMNN, Section 5.3]. Lemma 5.13 needs some background input from equivariant stable homotopy theory. Finish with Prop. 5.17.

**Talk 11: 16.07.2019 – Examples: Further Galois extensions II.**
Continue with [CMNN, Section 5.3] and then go on to explain the Galois extensions of $TMF$ that arise from étale and separated morphisms of Deligne–Mumford stacks $X \to \mathcal{M}_{ell}$ as in [CMNN, Example 5.26].

**Talk 12: 23.07.2019 – Examples: Further and non-Galois extensions.**
Continue with [CMNN, Theorem 5.27]. Finally, explain [CMNN, Section 5.4] about the connective versions of previous examples (which are not Galois in Rognes' sense but still satisfy descent).

## References

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[CMNN] D. Clausen, A. Mathew, N. Naumann, J. Noel: *Descent in algebraic $K$-theory and a conjecture of Ausoni–Rognes* to appear in J. Europ. Math. Soc.

[DHS] E. Devinatz, M. Hopkins, J. Smith: *Nilpotence and stable homotopy theory. I* Ann. of Math. (2), 128(2):207–241, 1988.

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[KN] A. Krause and T. Nikolaus: *Lectures on topological Hochschild homology and cyclotomic spectra* available at https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/papers.html

[Lur] J. Lurie: *Chromatic homotopy theory lecture notes.* available at author's homepage: http://www.math.harvard.edu/ lurie/252x.html

[MNN15] A. Mathew, N. Naumann, J. Noel: *On a nilpotence conjecture of J. P. May.* J. Topol., 8(4):917–932, 2015.

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[Mit] S. A. Mitchell: *The Morava K-theory of algebraic K-theory spectra.* K-Theory, 3(6):607?626, 1990.

[T] R. W. Thomason: *Algebraic K-theory and étale cohomology* Ann. Sci. Ecole Norm. Sup. (4), 18(3):437–552, 1985.

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