LECTURES ON HERMITIAN K-THEORY

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Abstract. These are lecture notes about a series of papers on hermitian K-theory, with main references [CDH+20a, CDH+20b, CDH+20c, CDH+20a, HL21, HLN, Lan]. The idea is to give a summary of the theory and the main results. The first goal is to give an overview over the structural properties of various forms of categories (stable, hermitian, Poincaré) and a discussion of the most important examples (module categories and categories of parametrized objects). Next, we give an overview of the proof of the fundamental sequence, relating Grothendieck–Witt theory to algebraic K-theory and L-theory. We then give an overview of multiplicative structures and categories of noncommutative Poincaré motives. Finally, we explain how to use the theory to perform calculations: Amongst other things, we will prove a general form of the Bass–Heller–Swan decomposition specialising to the fundamental theorem of K-theory, and the Shaneson–Ranicki splitting in L-theory. We use this to describe the Karoubi localisation of Grothendieck–Witt theory and L-theory and their relation to ordinary Grothendieck–Witt theory and L-theory. Finally, we discuss the Grothendieck–Witt theory of Dedekind rings, and some other natural examples arising in the theory.

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1. Poincaré categories

1.1. A short recap on stable ∞-categories.

Definition 1.1. Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is called stable if it is pointed, admits finite limits and a square is a pullback if and only if it is a pushout. A functor between stable categories is called exact if We denote by $\mathrm{Cat}^{\mathrm{ex}}_\infty$ the subcategory of Cat_∞ whose objects are stable categories and whose functors are exact.

Exercise. Let $\mathcal{C}$ be a pointed category such that each morphism has a fibre and a cofibre, and such that fibre sequences are cofibre sequences, and vice versa. Show that $\mathcal{C}$ is stable.

Example 1.2. The terminal category $*$ and the category of spectra Sp are stable. More generally $\mathrm{Mod}(R)$ for $R \in \mathrm{Alg}(\mathrm{Sp})$[1], $\mathrm{Perf}(R) = \mathrm{Mod}(R)^\omega$, $\mathrm{Fun}(K, \mathcal{C})$ for K an ∞-category and $\mathcal{C}$ a stable ∞-category, $\mathcal{C}^{\mathrm{op}}$, for $\mathcal{C}$ a stable category. Products of stable categories are stable, more generally pullbacks along exact functors between stable categories are again stable.

Remark 1.3. In general, for a category with finite limits $\mathcal{C}$, one can form pointed objects $\mathcal{C}_*$ and then take a limit over Ω (in Cat_∞) The resulting category can then be shown to be stable, and is called spectrum objects in $\mathcal{C}$, denoted $\mathrm{Sp}(\mathcal{C})$. As a direct step one can instead consider reduced and excisive functors $\mathrm{Fun}^{\mathrm{red,exc}}(\mathrm{An}_*, \mathcal{C})$ which can be shown to be equivalent to $\mathrm{Sp}(\mathcal{C})$.

Fact 1.4. There is an adjunction $\Sigma^\infty_+ \colon \mathrm{An} \rightleftarrows \mathrm{Sp} \colon \Omega^\infty$, and a unique symmetric monoidal structure $\otimes$ on Sp which commutes with colimits in each variable separately, and for which Σ^∞_+ refines to a symmetric monoidal functor. The tensor unit of the resulting symmetric monoidal structure is therefore the sphere spectrum $\mathbb{S} = \Sigma^\infty_+ *$.

Lemma 1.5. *Let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory and $\mathcal{C}$ stable. If $\mathcal{D}$ is closed under finite limits and finite colimits in $\mathcal{C}$, then $\mathcal{D}$ is stable.*

Remark 1.6. In fact, it suffices that $\mathcal{D}$ contains a zero object and is closed under the formation of fibres and suspensions (or dually under the formation of cofibres and loops).

[1] We shall always mean right modules when we write $\mathrm{Mod}(-)$, and notice that $\mathrm{Mod}(\mathbb{S}) \simeq \mathrm{Sp}$ and $\mathrm{Mod}(0) \simeq *$.

Fact 1.7. There is a lax symmetric monoidal functor $(\mathrm{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z}) \to (\mathrm{Sp}, \otimes_{\mathbb{S}}, \mathbb{S})$ called the Eilenberg–Mac Lane functor. Therefore, discrete rings can be viewed as objects of $\mathrm{Alg}(\mathrm{Sp})$, i.e. ring spectra. We will denote both objects simply by R and refrain from writing HR. Then we find that $\mathrm{Mod}(R) \simeq \mathcal{D}(R)$, i.e. the category of modules over the ring spectrum R is equivalent to the usual derived category of R[2]. This can be obtained from the 1-category $\mathrm{Ch}(R)$ of chain complexes over R by formally inverting quasi isomorphisms. See [Nik20] for a treatment. This perspective is important for us when relating L-theory in our setup to Ranicki's L-theory. It can also be obtained from $\mathrm{Proj}(R)$ by freely adjoining sifted colimits[3] and desuspensions. This perspective is important for us when relating GW-theory in our setup to the classical one (obtained via group completions).

Proposition 1.8. *Let $\mathcal{C}$ be a stable ∞-category, and let $\Omega^\infty \colon \mathrm{Sp} \to \mathrm{An}$ be the canonical functor. Then there exists a unique dashed arrow which is exact in each variable, making the triangle*

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "\mathrm{Map}(-,-)"] \arrow[dr, dashed, bend right, "\mathrm{map}(-,-)"'] & \mathrm{An} \\ & \mathrm{Sp} \arrow[u, "\Omega^\infty"'] \end{tikzcd}$$

commute, where the top horizontal arrow is the bivariant mapping anima functor.

Proof. This follows from the fact that for every stable category $\mathcal{C}$, $\Omega^\infty \colon \mathrm{Sp} \to \mathrm{An}$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathrm{Sp}) \longrightarrow \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{An})$$

using that $\mathrm{Map}_{\mathcal{C}}(-,-)$ is an object of $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{An}))$. □

In other words, the mapping anima in a stable category canonically refine to spectra, functorially in both variables.

Finally, we will need to distinguish small from and certain large stable ∞-categories.

Definition 1.9. An ∞-category $\mathcal{C}$ is called presentable if it is accessible and cocomplete. A presentable category $\mathcal{C}$ is called compactly generated if it is ω-accessible, i.e. if $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}^\omega)$, where $\mathcal{C}^\omega$ denotes the compact objects of $\mathcal{C}$, i.e. those X where $\mathrm{Map}_{\mathcal{C}}(X, -) \colon \mathcal{C} \to \mathrm{An}$ preserves (ω)-filtered colimits.

We say that a stable category $\mathcal{C}$ is compactly generated by a set of objects S if S is a set of compact objects such that the localising subcategory generated by S is equal to $\mathcal{C}$, or equivalently if S consists of compact objects such that the functor $\prod_{x \in S} \mathrm{Map}_{\mathcal{C}}(x, -) \colon \mathcal{C} \to \mathrm{Sp}$ is conservative.

Example 1.10. Let $\mathcal{C}$ be a small stable category. Then $\mathrm{Ind}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ is a presentable, compactly generated stable ∞-category. The Yoneda functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ is fully faithful and exact. The image of the Yoneda embedding lies in the ω-compact objects $\mathrm{Ind}(\mathcal{C})^\omega$ of $\mathrm{Ind}(\mathcal{C})$ and identifies $\mathrm{Ind}(\mathcal{C})^\omega$ as the idempotent completion $\mathcal{C}^\natural$ of $\mathcal{C}$.

Proposition 1.11. *The association $\mathcal{C} \mapsto \mathcal{C}^\natural$ extends to a left adjoint of the inclusion $\mathrm{Cat}_\infty^{\mathrm{perf}} \subseteq \mathrm{Cat}_\infty^{\mathrm{ex}}$.*

[2]This can be seen as an instance of the Schwede-Shipley theorem.

[3]This is a Dold-Kan type theorem.

Proposition 1.12. *There is an equivalence of categories*

$$\mathrm{Ind}\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \rightleftarrows \mathrm{Pr}_\omega^{\mathrm{st}} \colon (-)^\omega$$

where $\mathrm{Pr}_\omega^{\mathrm{st}}$ *denotes the category of compactly generated, presentable, stable* ∞*-categories with functors which preserve colimits and compact objects.*

Example 1.13. Let $R \in \mathrm{Alg}(\mathrm{Sp})$. Then $\mathrm{Mod}(R)$ is a presentable stable category, compactly generated by the object R, i.e. R is compact and the smallest stable subcategory of $\mathrm{Mod}(R)$ which is closed under colimits and contains R is $\mathrm{Mod}(R)$. This smallest subcategory is often called the *localising* subcategory generated by R. Likewise, the smallest stable subcategory (i.e. automatically closed under finite colimits) closed under retracts (the *thick* subcategory generated by R) is $\mathrm{Perf}(R)$. Finally, the smallest stable subcategory containing R (this does not have a name, to the best of my knowledge) is written dFree, I'll called it derived free to distinguish it from the 1-category of free modules for now. In fact, a discrete module lies in dFree if it is *stably* free already.

We remark that objects of $\mathrm{Perf}(R)$ can be modelled by finite chain complexes of finitely generated projectives, and objects of $\mathrm{dFree}(R)$ can be modelled by finite chain complexes of finitely generated free modules.

More generally, Thomason showed that there is a bijective correspondence between

$$\Big\{ \text{ subgroups } c \text{ of } K_0(\mathcal{C}) \Big\} \cong \Big\{ \text{ dense subcategories } \mathcal{C}_c \text{ of } \mathcal{C} \Big\}.$$

Here, a dense subcategory is one whose inclusion induces an equivalence after passing to idempotent completions. There is a canonical map from right to left sending $\mathcal{C}_c$ to the image of the map $K_0(\mathcal{C}_c) \to K_0(\mathcal{C})$ induced by the inclusion.

With this notation, the category $\mathrm{dFree}(R)$ is the subcategory which corresponds to the subgroup given by the image of the map $\mathbb{Z} \to K_0(R)$ which sends 1 to R. For a subgroup $c \subseteq K_0(R)$[4] we will write $\mathrm{Perf}^c(R)$ for the corresponding subgroup.

Exercise. Find concrete generators of the dense subcategory corresponding to $0 \subseteq K_0(\mathcal{C})$.

Example 1.14. Let $X \in \mathrm{An}$ be an anima. Then $\mathrm{Fun}(X, \mathrm{Mod}(R))$ is a presentable stable category, compactly generated by the objects $i_!(R)$ where $i\colon \{x\} \to X$ for each $x \in \pi_0(X)$. We obtain two small stable ∞-categories (in fact idempotent complete) by passing to compact objects: On the one hand side we have $\mathrm{Fun}(X, \mathrm{Perf}(R))$ and on the other hand we have $\mathrm{Fun}(X, \mathrm{Mod}(R))^\omega$.

Exercise. Work out under what conditions on X one has the following inclusions (inside $\mathrm{Fun}(X, \mathrm{Mod}(R))$):

(i) $\mathrm{Fun}(X, \mathrm{Mod}(R))^\omega \subseteq \mathrm{Fun}(X, \mathrm{Perf}(R))$, and
(ii) $\mathrm{Fun}(X, \mathrm{Perf}(R)) \subseteq \mathrm{Fun}(X, \mathrm{Mod}(R))^\omega$.

Remark 1.15. As an aside, let G be a finite group. Then one has a fully faithful inclusion $\mathrm{Fun}(BG, \mathrm{Mod}(R))^\omega \subseteq \mathrm{Fun}(BG, \mathrm{Perf}(R))$. The Verdier quotient, i.e. the quotient in $\mathrm{Cat}_\infty^{\mathrm{ex}}$, is called the stable module category of G and R. Sometimes also its idempotent completion is called the stable module category. For a discrete ring R which is regular coherent, the Verdier quotient is idempotent complete if and only if $K_{-1}(RG) = 0$ (we might come to this later).

[4]The c stands for "control", in that we control the objects in the category through their K-theory class.

1.2. **Reduced and 2-excisive functors between stable categories.** We will deal with contravariant functors throughout this lecture. This is why we also choose to make the basic definitions for contravariant functors.

Definition 1.16. A functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ between stable categories is called reduced if it sends zero objects to zero objects. In that case, one finds that $FX \oplus FY$ to be a functorial direct summand of $F(X \oplus Y)$ whose complement will be denoted by $B_F(X, Y)$. More concretely, we have

$$B_F(X, Y) \simeq \mathrm{cofib}(FX \oplus FY \to F(X \oplus Y)) \simeq \mathrm{fib}(F(X \oplus Y) \to FX \oplus FY)$$

Definition 1.17. We let

(i) $\mathrm{Fun}^{\mathrm{red}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ be the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ on reduced functors,
(ii) $\mathrm{Fun}^{\mathrm{bired}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathcal{D})$ on functors which are reduced in each variable,
(iii) $\mathrm{Fun}^{\mathrm{sbired}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) = \mathrm{Fun}^{\mathrm{bired}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})^{hC_2}$ the category of symmetric bireduced functors,
(iv) $\mathrm{Fun}^{\mathrm{b}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ be the full subcategory of $\mathrm{Fun}^{\mathrm{bired}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ which are bilinear, i.e. linear = exact in variable,
(v) $\mathrm{Fun}^{\mathrm{sb}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) = \mathrm{Fun}^{\mathrm{b}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})^{hC_2}$ the category of *symmetric* bilinear functors.

Remark 1.18. There are pullback diagrams

$$\begin{array}{ccccc}
\mathrm{Fun}^{\mathrm{sb}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) & \longrightarrow & \mathrm{Fun}^{\mathrm{sbired}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathcal{D})^{hC_2} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Fun}^{\mathrm{b}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) & \longrightarrow & \mathrm{Fun}^{\mathrm{bired}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathcal{D})
\end{array}$$

In particular, the top horizontal arrows are again fully faithful. Therefore, a symmetric bilinear functor is the same thing as a symmetric functor whose underlying functor of 2 variables is bilinear, and same with symmetric bireduced functors.

Lemma 1.19. *For a reduced functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, we have that $B_F \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathcal{D})^{hC_2}$, i.e. B is symmetric in its two input variables. Consequently, $B \circ \Delta \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})^{hC_2} \simeq \mathrm{Fun}(BC_2, \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}(BC_2, \mathcal{D}))$. Furthermore, B_F is reduced in each variable separately. In addition, the natural maps*

$$B_F(X, X) \longrightarrow F(X \oplus X) \xrightarrow{\Delta^*} F(X) \xrightarrow{\nabla^*} F(X \oplus X) \longrightarrow B(X, X)$$

are equivariant, and hence induce a canonical maps

$$B_F(X, X)_{hC_2} \longrightarrow F(X) \longrightarrow B_F(X, X)^{hC_2}$$

whose composite is the norm of the C_2-object $B_F(X, X)$.

Proof. Only the statement about the norm needs further justification, for the argument see e.g. [CDH+20a, 1.3.8] □

Definition 1.20. The association $B_-\colon \mathrm{Fun}^{\mathrm{red}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{bired}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ is called the cross effect. The association $B_-\colon \mathrm{Fun}^{\mathrm{red}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{sbired}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ is called the symmetric cross effect.

We now recall that a functor $\mathcal{Q}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is 2-excisive if it sends strongly cocartesian 3-cubes to cartesian 3-cubes. For our purposes, it is convenient to reformulate this condition as follows[5].

Proposition 1.21. *A reduced functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is 2-excisive if and only if the following conditions are satisfied:*

(i) *the cross effect B_F is bilinear, i.e. lies in the full subcategory* $\mathrm{Fun}^{\mathrm{b}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \subseteq \mathrm{Fun}^{\mathrm{bired}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$,
(ii) *the cofibre Λ_F of the map $(B_F \circ \Delta)_{hC_2} \to F$ is an exact functor.*

We will denote the category of reduced 2-excisive functors by $\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$[6]. Therefore, by definition and Remark 1.18, there is a functor

$$B\colon \mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \longrightarrow \mathrm{Fun}^{\mathrm{sb}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}).$$

From a reduced functor, one can also construct a linear functor as follows.

Definition 1.22. For a reduced functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ between stable categories, we define its excisive (or linear) approximation $P_1F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ as the colimit of the sequence[7]

$$F \longrightarrow \Omega F(\Omega -) \longrightarrow \Omega^2 F(\Omega^2 -) \longrightarrow \dots$$

Exercise. Show that P_1F is indeed and exact functor between stable categories.

Remark 1.23. The functor $P_1\colon \mathrm{Fun}^{\mathrm{red}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{exc}}(\mathcal{C}, \mathcal{D})$ is a left adjoint to the inclusion $\mathrm{Fun}^{\mathrm{exc}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}^{\mathrm{red}}(\mathcal{C}, \mathcal{D})$ and therefore preserves cofibre sequences of reduced functors[8].

Remark 1.24. Note that the cross effect of an exact functor vanishes. Therefore, for any reduced functor F, we obtain a natural commutative diagram

$$\begin{tikzcd} B_F(X,X)_{hC_2} \arrow[r] \arrow[d] & F \arrow[d] \\ 0 \arrow[r] & P_1(F) \end{tikzcd}$$

Proposition 1.25. *For a reduced and 2-excisive functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ the square of Remark 1.24 is exact. In other words, if Λ_F is exact, it is equivalent to P_1F.*

Proof. We need to show that the map $F \to \Lambda_F$ exhibits the target as the initial linear functor under F, i.e. that for any linear functor $G\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, we have that the restriction map

$$\mathrm{map}(\Lambda_F, G) \longrightarrow \mathrm{map}(F, G)$$

is an equivalence. Since $\mathrm{Fun}^{\mathrm{red}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ is stable, this is equivalent to showing that

$$\mathrm{map}((B_F \circ \Delta)_{hC_2}, G) = 0.$$

Now we note:

[5] For most, if not all, this can simply serve as a definition for the purpose of this course.

[6] In [CDH+20a] these are called quadratic functors and are denoted by $\mathrm{Fun}^{\mathrm{q}}(\mathcal{C})$. I feel there are too many quadratic things around, so I will stick to reduced 2-excisive.

[7] Note that we are forming Ω in $\mathcal{C}$ and $\mathcal{D}$, or equivalently Σ in $\mathcal{C}^{\mathrm{op}}$ and Ω in $\mathcal{D}$, as would be the usual formula for the excisive approximation.

[8] Note that $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}^{\mathrm{red}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are closed under fibres and cofibres and contains the zero object, and is hence are stable subcategories of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ by Lemma 1.5

Lemma 1.26. *There are adjunctions*

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{bired}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathcal{D}) \arrow[r, "\Delta^*"] & \mathrm{Fun}^{\mathrm{red}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \arrow[l, bend right, "B"'] \arrow[l, bend left, "B"] \end{tikzcd}$$

exhibiting B as both left and right adjoint to Δ^*.

Proof of Lemma 1.26. The transformations $B_F \circ \Delta \Rightarrow F \Rightarrow B_F \circ B_F$ are counit and unit of adjunctions exhibiting B as left and right adjoint to Δ^*. $\square$

We now note that Δ^* is canonically C_2-equivariant. Consequently, the above adjunctions refine canonically to C_2-equivariant adjunctions (i.e. the adjoints are also equivariant, in a way making the unit and counit equivariant transformations). Therefore, one obtains another such adjunction diagram after taking C_2-fixed points

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{bired}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathcal{D})^{hC_2} \arrow[r, "\Delta^*"] & \mathrm{Fun}(BC_2, \mathrm{Fun}^{\mathrm{red}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})) \arrow[l, bend right, "B"'] \arrow[l, bend left, "B"] \end{tikzcd}$$

We deduce that

$$\begin{aligned} \mathrm{map}_{\mathrm{Fun}^{\mathrm{red}}(\mathcal{C}^{\mathrm{op}},\mathcal{D})}((B_F \circ \Delta)_{hC_2}, G) &\simeq \mathrm{map}_{\mathrm{Fun}^{\mathrm{red}}(\mathcal{C}^{\mathrm{op}},\mathcal{D})^{BC_2}}(B_F \circ \Delta, G) \\ &\simeq \mathrm{map}_{\mathrm{Fun}^{\mathrm{bired}}(\mathcal{C}^{\mathrm{op}}\times\mathcal{C}^{\mathrm{op}},\mathcal{D})^{hC_2}}(B_F, B_G) \\ &= 0 \end{aligned}$$

For the last equation, it suffices to know that B_G is trivial in $\mathrm{Fun}^{\mathrm{bired}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathcal{D})$ since the forgetful functor $\mathrm{Fun}^{\mathrm{bired}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathcal{D})^{hC_2} \to \mathrm{Fun}^{\mathrm{bired}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathcal{D})$ is conservative. But now we use again that the cross effect of an exact functor vanishes. $\square$

Exercise. Let $\mathcal{C}$ be a category with G-action. Show that the functor $\mathcal{C}^{hG} \to \mathcal{C}$ is conservative.

Remark 1.27. The argument above shows that the symmetric cross effect functor, which is the composite

$$\mathrm{Fun}^{\mathrm{red}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \xrightarrow{\mathrm{const}} \mathrm{Fun}(BC_2, \mathrm{Fun}^{\mathrm{red}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})) \xrightarrow{B^{hC_2}} \mathrm{Fun}^{\mathrm{bired}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathcal{D})^{hC_2}$$

has right adjoint given by $(-)^{hC_2} \circ \Delta^*$ and left adjoint given by $(-)_{hC_2} \circ \Delta^*$. In particular, the formation of the symmetric cross effect, as well as the formation of the cross effect (i.e. forgetting the C_2-equivariance of the symmetric cross effect) preserves limits and colimits.

Corollary 1.28. *For a reduced a 2-excisive functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, there is a canonical pullback diagram in* $\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$

$$\begin{tikzcd} F \arrow[r, "P_1"] \arrow[d] & \Lambda_F \arrow[d] \\ (B_F \circ \Delta)^{hC_2} \arrow[r] & (B_F \circ \Delta)^{tC_2} \end{tikzcd}$$

Proof. First, we check that all terms are reduced 2-excisive:

Lemma 1.29. *Let $B \in \mathrm{Fun}^{\mathrm{sb}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$. Then $(B \circ \Delta)^{hC_2}$ is reduced 2-excisive and the cross effect of $(B \circ \Delta)^{hC_2}$ is equivalent to B. In particular, $(B \circ \Delta)^{tC_2}$ is exact.*

Proof. We note again that $\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ is closed under colimits and limits. Therefore, it suffices to show that $B \circ \Delta$ is reduced 2-excisive. One calculates that its symmetric cross effect is given by the induced C_2-object $B[C_2]$ in $\mathrm{Fun}^{\mathrm{sb}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$. Moreover, the transformation $(B[C_2] \circ \Delta)_{hC_2} \to B \circ \Delta$ is an equivalence. In particular, its cofibre is exact. We deduce that $B \circ \Delta$ is reduced 2-excisive. Now, the functor B also commutes with limits and colimits, we deduce that the cross effect of $(B \circ \Delta)^{hC_2}$ is equivalent to $B[C_2]^{hC_2} = B$. □

Now, to see that the above square is a pullback, we observe that both horizontal fibres are $(B_F \circ \Delta)_{hC_2}$. Therefore, it suffices to note that

$$\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$$

is a stable subcategory: it is in fact closed under limits and colimits as follows for instance from the characterisation of sending strongly cocartesian cubes to cartesian cubes. □

Remark 1.30. The argument in this proof can also be used to show that the canonical map $(B \circ \Delta)_{hC_2} \to (B \circ \Delta)^{hC_2}$ appearing in Lemma 1.19 is the norm map. This is the approach taken in [CDH+20a, 1.3.8]

Exercise. Show by hand that $(B \circ \Delta)^{tC_2}$ is exact.

In fact, one can upgrade the pointwise pullback of Corollary 1.28 to a statement on the level of categories:

Corollary 1.31. *There is a pullback diagram*

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \arrow[r] \arrow[d, "B"] & \mathrm{Fun}(\Delta^1, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})) \arrow[d, "t"] \\
\mathrm{Fun}^{\mathrm{sb}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \arrow[r] & \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})
\end{tikzcd}$$

of stable categories, where the lower horizontal functor sends B to $(B \circ \Delta)^{tC_2}$.

Proof. The diagram commutes, so one obtains a canonical functor from $\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ to the pullback. We will show that this comparison functor is essentially surjective and fully faithful. For essential surjectivity, consider a symmetric bilinear functor B and a linear functor G equipped with a map $G \to (B \circ \Delta)^{tC_2}$. Define F as the pullback

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & G \arrow[d] \\
(B \circ \Delta)^{hC_2} \arrow[r] & (B \circ \Delta)^{tC_2}
\end{tikzcd}$$

Since $\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ is closed under pullbacks in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$, we deduce from Lemma 1.29 and the fact that linear functors are reduced 2-excisive that F is reduced 2-excisive as well. We then claim that this diagram identifies with the canonical diagram appearing in Corollary 1.28. Since the functor B is exact and vanishes on linear functors, we deduce from Lemma 1.29 that B_F is equivalent to B. We deduce that $G = \Lambda_F$, so fully faithfulness is obtained. To see that it is also fully faithful, let F and G be reduced 2-excisive functors. We then obtain a fibre pullback

$$\begin{tikzcd}
\mathrm{map}(F,G) \arrow[r] \arrow[d] & \mathrm{map}(F,\Lambda_G) \arrow[d] \\
\mathrm{map}(F,(B_G\circ\Delta)^{hC_2}) \arrow[r] & \mathrm{map}(F,(B_G\circ\Delta)^{tC_2})
\end{tikzcd}$$

As before, we have equivalences $\mathrm{map}(F,(B_G\circ\Delta)^{hC_2}) \simeq \mathrm{map}(B_F,B_G)$ and $\mathrm{map}(F,\Lambda_G) \simeq \mathrm{map}(\Lambda_F,\Lambda_G)$. We therefore obtain that in the diagram

$$\begin{tikzcd}
\mathrm{map}(F,G) \arrow[r] \arrow[d] & \mathcal{X} \arrow[r] \arrow[d] & \mathrm{map}(\Lambda_F,\Lambda_G) \arrow[d] \\
\mathrm{map}(B_F,B_G) \arrow[r] & \mathrm{map}((B_F\circ\Delta)^{tC_2},(B_F\circ\Delta)^{tC_2}) \arrow[r] & \mathrm{map}(\Lambda_F,(B_G\circ\Delta)^{tC_2}
\end{tikzcd}$$

the big square is a pullback. Defining $\mathcal{X}$ as the pullback of the right square, we find that

$$\mathcal{X} \simeq \mathrm{map}_{\mathrm{Fun}(\Delta^1,\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}},\mathcal{D}))}((\Lambda_F\to(B_F\circ\Delta)^{tC_2}),(\Lambda_G\to(B_G\circ\Delta)^{tC_2}))$$

so that the left pullback square shows the required fully faithfulness. □

1.3. Hermitian and Poincaré structures.

Definition 1.32. Let $\mathcal{C}$ be a stable category. A hermitian structure on $\mathcal{C}$ is a reduced functor 2-excisive functor $\Omega\colon \mathcal{C}^{\mathrm{op}}\to\mathrm{Sp}$. A stable category equipped with a hermitian structure is called a hermitian category.

Next we wish to define Poincaré structures and Poincaré categories. These will be hermitian structures which are required to satisfy a certain non-degeneracy condition. To explain it, we note that given a symmetric bilinear functor B, we may view B as an object of

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}},\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}},\mathrm{Sp})).$$

It may or may not lie in the full subcategory

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}},\mathcal{C}) \subseteq \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}},\mathrm{Sp}))$$

induced by the Yoneda embedding $\mathcal{C}\subseteq\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}},\mathrm{Sp})$. If this is the case, we call B *non-degenerate.*

For a non-degenerate symmetric bilinear functor, we therefore obtain an essentially unique exact functor $D\colon\mathcal{C}^{\mathrm{op}}\to\mathcal{C}$, characterised by a natural equivalence

$$B(X,Y)\simeq\mathrm{map}_{\mathcal{C}}(X,DY),$$

which we refer to as the pre-duality associated to the non-degenerate functor B. By virtue of the symmetry of B, we obtain a canonical equivalence

$$\mathrm{map}_{\mathcal{C}^{\mathrm{op}}}(DX,Y)=\mathrm{map}_{\mathcal{C}}(Y,DX)\simeq B(Y,X)\simeq B(X,Y)\simeq\mathrm{map}_{\mathcal{C}}(X,DY)$$

which exhibits D (or more precisely D^{op}) as left and right adjoint to D. In particular, we obtain a canonical transformation $\mathrm{id}\Rightarrow D^2$. We call a symmetric bilinear functor B *perfect* if it is non-degenerate and the just described transformation $\mathrm{id}\Rightarrow D^2$ is an equivalence.

Remark 1.33. It follows from the fact that $\mathrm{id} \Rightarrow D^2$ is the unit of an adjunction that it is an equivalence if and only if D is itself an equivalence of categories. One can therefore rephrase that a non-degenerate symmetric bilinear functor is perfect if and only if its associated pre-duality is an equivalence, in which case we call it the associated duality.

Definition 1.34. A hermitian structure Ω on a stable category $\mathcal{C}$ is called Poincaré, if its cross effect B_{Ω} is perfect. A hermitian category whose hermitian structure is Poincaré is called a Poincaré category. In this case, we refer to D as the underlying duality of the Poincaré category.

Remark 1.35. Let $(\mathcal{C}, \Omega)$ be a hermitian (Poincaré) category. Then $(\mathcal{C}, \Sigma\Omega)$ and $(\mathcal{C}, \Omega\Omega)$ are again hermitian (Poincaré). The underlying dualities are given by ΣD and ΩD, respectively.

Example 1.36. Let $\mathcal{C}$ be a stable category and B a symmetric bilinear functor. Then the functors

$$\Omega^{\mathrm{s}}_B(X) = B(X,X)^{hC_2} \quad \text{and} \quad \Omega^{\mathrm{q}}_B(X) = B(X,X)_{hC_2}$$

are hermitian structures, whose cross effect are canonically equivalent to B. In particular, they are Poincaré if B is perfect. Ω^{s}_B is called the homotopy symmetric hermitian structure, and Ω^{q}_B is called the homotopy quadratic structure.

In terms of the canonical pullback square of Corollary 1.28, the linear part of Ω^{s}_B is given by $B(X,X)^{tC_2}$ and the linear part of Ω^{q}_B is 0. Each of these, of course, come with canonical maps to $B(X,X)^{tC_2}$.

Fact 1.37. We may consider the C_2-action on $\mathrm{Cat}^{\mathrm{ex}}_\infty$ given by sending $\mathcal{C}$ to $\mathcal{C}^{\mathrm{op}}$ and the category $(\mathrm{Cat}^{\mathrm{ex}}_\infty)^{hC_2}$. An object of this category is a stable category $\mathcal{C}$, equipped with a coherent symmetric duality $D\colon \mathcal{C}^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{C}$. It turns out that the datum of a coherent duality D on $\mathcal{C}$ is equivalent to the datum of a perfect symmetric bilinear functor. The correspondence, of course, being given by sending $(\mathcal{C}, D)$ to the functor $B(X,Y) = \mathrm{map}_{\mathcal{C}}(X, DY)$. See [CDH+20a, 7.2.15] for the details.

Remark 1.38. This shows that Poincaré categories are canonically stable categories with duality, and that every stable category with duality gives rise to at least two Poincaré structures with the same underlying stable category with duality: The homotopy symmetric and the homotopy quadratic one. Put more categorically, we find that there is a forgetful functor[9]

$$\mathrm{Cat}^{\mathrm{p}}_\infty \longrightarrow (\mathrm{Cat}^{\mathrm{ex}}_\infty)^{hC_2}$$

which turns out to admit a left, respectively right, adjoint, given by sending a category with duality to the homotopy quadratic, respectively homotopy symmetric, structure of its associated symmetric bilinear functor as in Example 1.36, see [CDH+20a, 7.2.16 & 7.2.17].

Example 1.39. Let R be a commutative ring and let $\mathrm{Perf}(R) = \mathcal{D}(R)^\omega$ be its category of perfect modules. The functor $\mathrm{Perf}(R)^{\mathrm{op}} \times \mathrm{Perf}(R)^{\mathrm{op}} \to \mathrm{Sp}$ given by sending (M,N) to $\mathrm{map}_R(M \otimes_R N, R)$ is a perfect symmetric bilinear functor[10] The fact that this functor is perfect comes from the fact that perfect R-modules are dualisable, so that the pre-duality $\mathrm{map}_R(-,R)\colon \mathrm{Perf}(R)^{\mathrm{op}} \to \mathrm{Perf}(R)$ is an equivalence as needed.

Therefore, for a commutative ring R, we obtain two canonical Poincaré structures Ω^{s}_R and Ω^{q}_R as a an application of Example 1.36.

[9] See Definition 1.44 for the definition of $\mathrm{Cat}^{\mathrm{p}}_\infty$.

[10] The symmetry comes from the fact that the relative tensor product $- \otimes_R -$ endows $\mathrm{Mod}(R)$ with the structure of a symmetric monoidal category.

We will later see that there are more Poincaré structures on $\mathrm{Perf}(R)$ than the homotopy symmetric and homotopy quadratic ones, and that there are generalisations of the above example to non-commutative rings R. In fact, in order to reproduce classical Grothendieck–Witt spectra for discrete rings in which 2 is not invertible, we will need to consider a Poincaré structure which is (in general) neither homotopy symmetric nor homotopy quadratic.

Example 1.40. Consider the functor $\Omega^{\mathrm{u}} \colon (\mathrm{Sp}^{\omega})^{\mathrm{op}} \to \mathrm{Sp}$ given by the following pullback

$$\begin{tikzcd} Ϙ^{\mathrm{u}}(X) \arrow[r] \arrow[d] & DX \arrow[d] \\ (DX \otimes DX)^{hC_2} \arrow[r] & (DX \otimes DX)^{tC_2} \end{tikzcd}$$

where D denotes the usual Spanier-Whitehead duality, i.e. $DX = \mathrm{map}(X, \mathbb{S})$, and where the right vertical map is given by the Tate diagonal [NS18][11]. Since $(DX \otimes DX) \simeq \mathrm{map}(X \otimes X, \mathbb{S}) \simeq \mathrm{map}(X, DX)$ we see that $Ϙ^u$ is perfect with underlying duality the Spanier-Whitehead duality. Furthermore $X \mapsto DX$ is clearly exact, so $Ϙ^{\mathrm{u}}$ is reduced 2-excisive as needed.

Remark 1.41. The Poincaré structure $Ϙ^{\mathrm{u}}$ is neither homotopy quadratic nor homotopy symmetric: Its linear part is neither trivial nor equivalent to $(DX \otimes DX)^{tC_2}$[12]

1.4. The categories $\mathrm{Cat}_\infty^{\mathrm{h}}$ and $\mathrm{Cat}_\infty^{\mathrm{p}}$.

Definition 1.42. We define the category of hermitian categories $\mathrm{Cat}_\infty^{\mathrm{h}}$ as the Grothendieck construction of the functor

$$(\mathrm{Cat}_\infty^{\mathrm{ex}})^{\mathrm{op}} \longrightarrow \mathrm{CAT}_\infty$$

given by sending $\mathcal{C}$ to $\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$.

We write

$$\mathrm{Cat}_\infty^{\mathrm{h}} = \int_{\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{ex}}} \mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$$

and note that therefore the canonical functor $\mathrm{Cat}_\infty^{\mathrm{h}} \to \mathrm{Cat}_\infty^{\mathrm{ex}}$, sending $(\mathcal{C}, Ϙ)$ to $\mathcal{C}$, is a cartesian fibration[13] Unravelling the definitions, a morphism in $\mathrm{Cat}_\infty^{\mathrm{h}}$ from a hermitian category $(\mathcal{C}, Ϙ)$ to another hermitian category $(\mathcal{C}', Ϙ')$ consists of an exact functor $f \colon \mathcal{C} \to \mathcal{D}$, and a natural transformation $\eta \colon Ϙ \Rightarrow f^*(Ϙ')$.

For a hermitian functor $(f, \eta) \colon (\mathcal{C}, Ϙ) \to (\mathcal{C}', Ϙ')$ between Poincaré categories, there is a canonical transformation $\tau_f \colon fD \Rightarrow D'f$[14]. To see this, we note that η induces a natural transformation $B_{Ϙ} \Rightarrow B_{Ϙ'}$. Therefore we obtain a canonical map

$$\mathrm{map}_{\mathcal{C}}(X, X) \simeq B_{Ϙ}(DX, X) \to B_{Ϙ'}(fDX, fX) \simeq \mathrm{map}_{\mathcal{C}'}(fDX, D'fX)$$

The identity on the left hand hence gives rise to a map $fDX \to D'fX$, and one can show that this map is the component of a natural transformation τ as needed.

[11]The Tate diagonal is a natural transformation $\mathrm{id} \Rightarrow (\otimes \circ \Delta)^{tC2}$, which is by the Yoneda lemma given by a point in $\mathbb{S}^{tC_2}$. Then, one takes the canonical map $\mathbb{S} \to \mathbb{S}^{hC_2} \to \mathbb{S}^{tC_2}$.

[12]I recommend to work out why DX is not equivalent to $(DX \otimes DX)^{tC_2}$, for instance for $X = \mathbb{S}$. Hint: Show that $\mathbb{S}^{tC_2}$ is 2-complete, and that $\mathbb{S}$ is not 2-complete.

[13]In fact, it is also a cocartesian fibration, essentially by left Kan extending hermitian structures, see [CDH+20a, Section 1.4].

[14]Again, I am ommiting op's at the functors

Definition 1.43. A morphism $(f, \eta)\colon (\mathcal{C}, \mathfrak{P}) \to (\mathcal{C}', \mathfrak{P}')$ between Poincaré categories is called a Poincaré functor if the transformation τ_f is an equivalence.

Definition 1.44. We let $\mathrm{Cat}^{\mathrm{p}}_\infty$ be the subcategory of $\mathrm{Cat}^{\mathrm{h}}_\infty$ whose objects are the Poincaré categories and whose morphisms are Poincaré functors.

Proposition 1.45. *The categories* $\mathrm{Cat}^{\mathrm{h}}_\infty$ *and* $\mathrm{Cat}^{\mathrm{p}}_\infty$ *are semi-additive, i.e. they are pointed, admit finite coproducts and products which are equivalent.*

Proof. Clearly $(0, 0)$ is both initial and terminal. Furthermore, it is not hard to see that for hermitian (Poincaré) categories $(\mathcal{C}, \mathfrak{P})$ and $(\mathcal{C}', \mathfrak{P}')$, the functor $\mathfrak{P} \oplus \mathfrak{P}'\colon \mathcal{C} \oplus \mathcal{C}' \to \mathrm{Sp}$ is a hermitian structure, which is Poincaré if $\mathfrak{P}$ and $\mathfrak{P}'$ are. In this case the underlying duality on $\mathcal{C} \oplus \mathcal{C}'$ is componentwise the given duality. It is not hard to see that this is in fact a product and a coproduct in $\mathrm{Cat}^{\mathrm{h}}_\infty$ ($\mathrm{Cat}^{\mathrm{p}}_\infty$). $\square$

1.5. **Hermitian and Poincaré objects.** Let $(\mathcal{C}, \mathfrak{P})$ be a hermitian category. We denote by $\mathrm{Herm}(\mathcal{C}, \mathfrak{P})$ its category of forms,

$$\mathrm{Herm}(\mathcal{C}, \mathfrak{P}) = \int_{\mathcal{C}} \Omega^\infty \mathfrak{P}$$

so that $\mathrm{Herm}(\mathcal{C}, \mathfrak{P}) \to \mathcal{C}$ is a right fibration classified by the functor $\Omega^\infty\mathfrak{P}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$. We then define the anima of forms on $(\mathcal{C}, \mathfrak{P})$ by the pullback

$$\begin{tikzcd} \mathrm{Form}(\mathcal{C}, \mathfrak{P}) \arrow[r] \arrow[d] & \mathrm{Herm}(\mathcal{C}, \mathfrak{P}) \arrow[d] \\ \mathrm{Core}(\mathcal{C}) \arrow[r] & \mathcal{C} \end{tikzcd}$$

Here, $\mathrm{Core}(\mathcal{C})$ denotes the maximal sub groupoid of $\mathcal{C}$, obtained from $\mathcal{C}$ by discarding all non-invertible morphisms. Since right fibrations are conservative, $\mathrm{Form}(\mathcal{C}, \mathfrak{P})$ is equivalently given by $\mathrm{Core}(\mathrm{Herm}(\mathcal{C}, \mathfrak{P}))$.

A point in $\mathrm{Form}(\mathcal{C}, \mathfrak{P})$ is therefore given by an object X equipped with a map $q\colon \mathbb{S} \to \mathfrak{P}(X)$. If $(\mathcal{C}, \mathfrak{P})$ is Poincaré, we may therefore consider the composite

$$\mathbb{S} \longrightarrow \mathfrak{P}(X) \longrightarrow B(X, X)^{hC_2} \longrightarrow B(X, X) \simeq \mathrm{map}_{\mathcal{C}}(X, DX)$$

which we will denote by $q_\sharp$.

Definition 1.46. A form (X, q) in a Poincaré category $(\mathcal{C}, \mathfrak{P})$ is called a Poincaré form (or a Poincaré object) if the map $q_\sharp$ is an equivalence.

Definition 1.47. For a Poincaré category $(\mathcal{C}, \mathfrak{P})$, we denote by $\mathrm{Poinc}(\mathcal{C}, \mathfrak{P})$ the collection of components of $\mathrm{Form}(\mathcal{C}, \mathfrak{P})$ on all Poincaré forms.

Proposition 1.48. *The associations* $(\mathcal{C}, \mathfrak{P}) \mapsto \mathrm{Form}(\mathcal{C}, \mathfrak{P})$ *and* $(\mathcal{C}, \mathfrak{P}) \mapsto \mathrm{Poinc}(\mathcal{C}, \mathfrak{P})$ *extend to functors* $\mathrm{Form}\colon \mathrm{Cat}^{\mathrm{h}}_\infty \to \mathrm{An}$ *and* $\mathrm{Poinc}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{An}$.

Proof. The functoriality of these constructions essentially follow from the functoriality properties of Grothendieck constructions, see [CDH+20a, 2.1.2 & 2.1.5] for an argument with references. Let us for now[15] only explain how a hermitian functor sends hermitian objects

[15] We will give another arugment for functoriality momentarily, see Lemma 1.50

to hermitian objects and why this map restricts to Poincaré objects if the functor was itself Poincaré.

So let $(f, \eta) \colon (\mathcal{C}, \Qoppa) \to (\mathcal{C}', \Qoppa')$ be a hermitian functor and let (X, q) be a hermitian form on $(\mathcal{C}, \Qoppa)$. Then the composite

$$\mathbb{S} \xrightarrow{q} \Qoppa(X) \xrightarrow{\eta} \Qoppa'(fX)$$

determines a form q' on fX. Now suppose that (f, η) is Poincaré. We then obtain that the map $q'_\sharp \colon fX \to D'fX$ is given by the composite

$$fX \xrightarrow{f(q_\sharp)} fDX \xrightarrow{\tau_f} D'fX.$$

The first map is an equivalence if (X, q) is a Poincaré object and the second map is an equivalence if (f, η) is Poincaré. Therefore, (fX, q') is a Poincaré object if (X, q) is and if (f, η) is a Poincaré functor. $\square$

Remark 1.49. For two hermitian categories $(\mathcal{C}, \Qoppa)$ and $(\mathcal{C}', \Qoppa')$, there is a canonical hermitian structure on $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{C}')$, denoted by $\mathrm{nat}^{\Qoppa}_{\Qoppa'}$. It sends a functor f to the spectrum of maps $\mathrm{map}_{\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})}(\Qoppa, f^*\Qoppa')$[16]. If $(\mathcal{C}, \Qoppa)$ and $(\mathcal{C}', \Qoppa')$ are Poincaré categories, then so is $(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{C}'), \mathrm{nat}^{\Qoppa}_{\Qoppa'})$. Moreover, we find that

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{h}}_\infty}((\mathcal{C}, \Qoppa), (\mathcal{C}', \Qoppa')) \simeq \mathrm{Form}((\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{C}'), \mathrm{nat}^{\Qoppa}_{\Qoppa'})$$

and that

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{p}}_\infty}((\mathcal{C}, \Qoppa), (\mathcal{C}', \Qoppa')) \simeq \mathrm{Poinc}(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{C}'), \mathrm{nat}^{\Qoppa}_{\Qoppa'}).$$

In other words, a Poincaré functor is indeed a Poincaré object in a suitable Poincaré category of functors. See [CDH+20a, 6.2.1, 6.2.2 & 6.2.12] for the details.

Lemma 1.50. *The functors* $\mathrm{Form} \colon \mathrm{Cat}^{\mathrm{h}}_\infty \to \mathrm{An}$ *and* $\mathrm{Poinc} \colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{An}$ *are corepresented by* $(\mathrm{Sp}^\omega, \Qoppa^{\mathrm{u}})$.

Proof. First, to make the statement precise, we have to exhibit a point in $\mathrm{Poinc}(\mathrm{Sp}^\omega, \Qoppa^{\mathrm{u}})$. The underlying object will be $\mathbb{S}$ and to construct the form $q^{\mathrm{u}}_{\mathbb{S}}$ we investigate the following diagram:

$$\begin{tikzcd} \Qoppa^{\mathrm{u}}(\mathbb{S}) \arrow[r] \arrow[d] & \mathbb{S} \arrow[d] \arrow[dl, dashed] \\ \mathbb{S}^{hC_2} \arrow[r] & \mathbb{S}^{tC_2} \end{tikzcd}$$

The dashed arrow makes the lower triangle commute, and hence gives rise to a section of the upper horizontal map. We claim that the resulting point determines a Poincaré structure on $\mathbb{S}$: For this, we need to consider the composite

$$\mathbb{S} \longrightarrow \Qoppa^{\mathrm{u}}(\mathbb{S}) \longrightarrow B_{\Qoppa^{\mathrm{u}}}(\mathbb{S}, \mathbb{S})^{hC_2} \longrightarrow B(\mathbb{S}, \mathbb{S}) \simeq \mathrm{map}(\mathbb{S}, D\mathbb{S}) = \mathbb{S}.$$

This composite is equivalent to the composition

$$\mathbb{S} \longrightarrow \mathbb{S}^{hC_2} \longrightarrow \mathbb{S}$$

which is indeed an equivalence.

[16] Recall that the category of reduced 2-excisive functors between stable categories is itself stable.

To see the corepresentability, we make use of Remark 1.49[17]. The decisive feature of the functor $Ϙ^{\mathrm{u}}$ is that it corepresents the functor $\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}((\mathrm{Sp}^\omega)^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Sp}$ given by evaluating on $\mathbb{S}$.[18] Now, we have that the functor $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp}^\omega, \mathcal{C}) \to \mathcal{C}$, given by evaluation on $\mathbb{S}$, is an equivalence of categories. Under this equivalence, the functor $\mathrm{nat}_{Ϙ}^{Ϙ^{\mathrm{u}}}$ is equivalent to $Ϙ$: We simply need to calculate the composite

$$\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp}^\omega, \mathcal{C})^{\mathrm{op}} \xrightarrow{\mathrm{ev}_{\mathbb{S}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{Ϙ} \mathrm{Sp}$$

which takes a functor f to $Ϙ(f(\mathbb{S}))$. However, by definition, we have

$$\mathrm{nat}_{Ϙ'}^{Ϙ^{\mathrm{u}}}(f) = \mathrm{map}_{\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathrm{Sp}^\omega,\mathrm{Sp})}(Ϙ^{\mathrm{u}}, f^*(Ϙ))$$

which is also equivalent to $Ϙ(f(\mathbb{S}))$ by the above corepresentability result for $Ϙ^{\mathrm{u}}$.

Therefore, we obtain an equivalence of hermitian (Poincaré) categories

$$(\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp}^\omega, \mathcal{C}), \mathrm{nat}_{Ϙ}^{Ϙ^{\mathrm{u}}}) \simeq (\mathcal{C}, Ϙ)$$

so that the corollary follows from Remark 1.49 once we have taken for granted that the last equivalence in Remark 1.49 sends the identity of $(\mathrm{Sp}^\omega, Ϙ^{\mathrm{u}})$, under the above equivalence

$$(\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp}^\omega, \mathrm{Sp}^\omega), \mathrm{nat}_{Ϙ^u}^{Ϙ^{\mathrm{u}}}) \simeq (\mathrm{Sp}^\omega, Ϙ^{\mathrm{u}})$$

to the form $q_{\mathbb{S}}^{\mathrm{u}}$ described above. □

Corollary 1.51. *The functors* $\mathrm{Form}\colon \mathrm{Cat}_\infty^{\mathrm{h}} \to \mathrm{An}$ *and* $\mathrm{Poinc}\colon \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathrm{An}$ *canonically lift along the forgetful map* $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{An}) \to \mathrm{An}$. *Concretely, the sum of two hermitian, respectively Poincaré objects* (X, q) *and* (X', q') *is given by* $X \oplus X'$ *with form*

$$(q, q') \in \Omega^\infty Ϙ(X) \times \Omega^\infty Ϙ(X') \longrightarrow \Omega^\infty Ϙ(X \oplus X').$$

Proof. Any product preserving functor induces between categories with finite products induces a functor on $\mathbb{E}_\infty$-monoids with respect to the cartesian product symmetric monoidal structure. Moreover, for such a category $\mathcal{C}$, one has that the forgetful functor $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathcal{C}$ is an equivalence if and only if $\mathcal{C}$ is semi-additive. Using that Form and Poinc are corepresented, we deduce that they preserve all limits and therefore products. The claim then follows from the semi-additivity of $\mathrm{Cat}_\infty^{\mathrm{h}}$ and $\mathrm{Cat}_\infty^{\mathrm{p}}$ Proposition 1.45. □

Let us consider the first standard examples of Poincaré objects.

Example 1.52. Let R be a commutative ring, P a finitely generated projective R-module, and φ a symmetric bilinear form on P. Concretely, $\varphi \in \mathrm{Hom}_R(P \otimes_R P, R)^{C_2}$. Viewing $P[0]$ as a perfect module in degree 0, we find that

$$\Omega^\infty Ϙ_R^{\mathrm{s}}(P) = \Omega^\infty \mathrm{map}_R(P[0] \otimes_R P[0], R)^{hC_2} \simeq \mathrm{Hom}_R(P \otimes_R P, R)^{C_2}$$

so $(P[0], \varphi)$ canonically determines a hermitian object for $(\mathrm{Perf}(R), Ϙ_R^{\mathrm{s}})$. This form is Poincaré if and only if φ is a unimodular form in the classical sense, i.e. is such that the adjunct map $P \to \mathrm{Hom}_R(P, R)$ is an isomorphism.

[17]This is going to lead to a slightly circular argument, but one can make a valid proof from the arguments presented here, see [CDH+20a, 4.1.3].

[18]It is a nice exercise to prove this fact.

Example 1.53. Let R be a commutative ring, P a finitely generate projective module and ψ a quadratic form on P, i.e. an element of $\mathrm{Hom}_R(P \otimes_R P, R)_{C_2}$[19]. In this case we find

$$\pi_0(\Omega^{\mathrm{q}}_R(P[0])) \simeq \pi_0(\mathrm{map}_R(P[0] \otimes_R P[0], R)_{hC_2}) \simeq \mathrm{Hom}_R(P \otimes_R P, R)_{C_2}$$

so that $(P[0], \psi)$ gives rise to a canonical hermitian object for $(\mathrm{Perf}(R), \Omega^{\mathrm{q}}_R)$. The norm map determines the polarisation of the quadratic form ψ, a symmetric bilinear form φ, and classically ψ is called unimodular if its polarisation φ is. Hence $(P[0], \psi)$ is a Poincaré object for $(\mathrm{Perf}(R), \Omega^{\mathrm{q}}_R)$ if ψ is a unimodular quadratic form on P in the classical sense.

Example 1.54. Let M be an oriented closed connected manifold of dimension d. Then $C^*(M)$ is canonically an object of $\mathrm{Perf}(\mathbb{Z})$. We claim that it refines to a Poincaré object for $(\mathrm{Perf}(\mathbb{Z}), \Omega^d \Omega^{\mathrm{s}}_{\mathbb{Z}})$. We note that evaluation on the fundamental class $[M]$ of M defines a map $C^*(M) \to \mathbb{Z}[-d]$, where $\mathbb{Z}[-d]$ is another notation for $\Omega^d \mathbb{Z}$ in $\mathrm{Perf}(\mathbb{Z})$. Now, since $\mathbb{Z}$ is a commutative ring, we find that $C^*(M)$ is an $\mathbb{E}_\infty$-algebra in $\mathrm{Perf}(\mathbb{Z})$. In particular, there is a C_2-equivariant multiplication map giving rise to the C_2-equivariant composite

$$C^*(M) \otimes_{\mathbb{Z}} C^*(M) \longrightarrow C^*(M) \longrightarrow \mathbb{Z}[-d]$$

where $\mathbb{Z}[-d]$ is equipped with the trivial C_2-action. This map determines a point q_M in the space

$$\Omega^\infty \mathrm{map}_{\mathbb{Z}}(C^*(M) \otimes_{\mathbb{Z}} C^*(M), \mathbb{Z}[-d])^{hC_2} = \Omega^\infty \Omega^d \Omega^{\mathrm{s}}_{\mathbb{Z}}(C^*(M)).$$

Unravelling the definitions, we find that the associated map $q_\sharp$ is given by the map

$$C^*(M) \longrightarrow C_*(M)[-d]$$

which is the cap product with the fundamental class of M. Therefore, by usual Poincaré duality, this map is an equivalence, and hence $(C^*(M), q_M)$ is indeed a Poincaré object of $(\mathrm{Perf}(\mathbb{Z}), \Omega^{\mathrm{s}}_{\mathbb{Z}})$.

Remark 1.55. This example can be generalised quite drastically. We will see that there is another Poincaré category $((\mathrm{Sp}_{/M})^\omega, \Omega^{\mathrm{v}}_{\nu_M})$ in which M determines a canonical Poincaré object, whenever M is a Poincaré duality space (a purely homotopy theoretic notion, of which closed connected manifolds are examples). When M is oriented and of dimension d, there is a Poincaré functor $((\mathrm{Sp}_M)^\omega, \Omega^{\mathrm{v}}_{\nu_M}) \to (\mathrm{Perf}(\mathbb{Z}), \Omega^d \Omega^{\mathrm{s}}_{\mathbb{Z}})$ which sends the Poincaré object associated to M to the Poincaré object $(C^*(M), q_M)$ constructed in Example 1.54

We will now consider some general types of forms: Let $(\mathcal{C}, \Omega)$ be a Poincaré category and let $X \in \mathcal{C}$ be an object. Consider the canonical map

$$\mathrm{map}(X, X) \longrightarrow \Omega(X \oplus DX)$$

which is the direct sum inclusion of the decomposition

$$\Omega(X \oplus DX) \simeq \Omega(X) \oplus \Omega(DX) \oplus \mathrm{map}(X, X).$$

The identity of X therefore determines a point q_X^{Hyp} of $\Omega^\infty \Omega(X \oplus DX)$.

Lemma 1.56. *Let $(\mathcal{C}, \Omega)$ be a Poincaré category and let $X \in \mathcal{C}$ be an object. Then the just constructed form q_X^{Hyp} is a Poincaré object.*

[19]It is a good exercise to work out that this definition of quadratic forms coincides with more classical versions, see e.g. [CDH+20a, 4.2.18]

Proof. Chasing through the definitions, we find that the associated map $X \oplus DX \to D(X \oplus DX) \simeq DX \oplus X$ is represented by the standard hyperbolic matrix

$$\begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

and is therefore invertible. $\square$

We call $(X \oplus DX, q_X^{\mathrm{Hyp}})$ the hyperbolic object associated to X. This construction can be categorified: Let $\mathcal{C}$ be a stable ∞-category. Let $\Omega^{\mathrm{Hyp}} = \mathrm{map}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$.

Lemma 1.57. *The pair $(\mathcal{C} \times \mathcal{C}^{\mathrm{op}}, \Omega^{\mathrm{Hyp}})$ is a Poincaré category. Its underlying duality is given by $(X,Y) \mapsto (Y,X)$. The identity of an object X gives rise to a canonical point of $\Omega^{\mathrm{Hyp}}(X,X)$ which makes (X,X) a Poincaré object.*

Proof. See [CDH+20a,]. $\square$

We write $\mathrm{Hyp}(\mathcal{C})$ for this Poincaré category and call it the *hyperbolic category.*

Remark 1.58. If $(\mathcal{C}, \Omega)$ is Poincaré, then there is a canonical hermitian structure on the functor $\mathrm{Hyp}(\mathcal{C}) \to (\mathcal{C}, \Omega)$ given by sending (X,Y) to $X \oplus DY$: It is the canonical inclusion

$$\Omega^{\mathrm{Hyp}}(X,Y) = \mathrm{map}_{\mathcal{C}}(X,Y) \simeq B(X, DY) \longrightarrow \Omega(X \oplus DY).$$

This functor is in fact Poincaré and sends the Poincaré object (X,X) of Lemma 1.57 to the hyperbolic object $(X \oplus DX, q_X^{\mathrm{Hyp}})$ of Lemma 1.56.

Finally, we explain the notions of Lagrangians in our setup. We recall that in algebra, a Lagrangian of a form φ on a finitely generated projective module P is a finitely generated projective submodule $L \subseteq P$ which is isotropic, i.e. on which φ vanishes, and such that the sequence

$$0 \longrightarrow L \longrightarrow P \cong DP \longrightarrow DL \longrightarrow 0$$

is exact.

In addition, a form (P, φ) is called *metabolic* if it admits a Lagrangian. Examples are the standard hyperbolic forms $(P \oplus DP, q^{\mathrm{Hyp}})$ where the inclusion $P \to P \oplus DP$ is a Lagrangian. However, there are also non-hyperbolic forms which admit a Lagrangian: For instance the form

$$\left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right)$$

has the inclusion $\mathbb{Z} \to \mathbb{Z}^2$ as first basis vector as a Lagrangian, but it is not isomorphic to the 2-dimensional hyperbolic form over $\mathbb{Z}$[20].

We adapt these definition to our setup as follows. Let $(\mathcal{C}, \Omega)$ be a Poincaré category and (X,q) a Poincaré form.

Definition 1.59. A Lagrangian of (X,q) consists of a map $f \colon L \to X$ together with a null-homotopy η of $f^*(q)$. The null-homotopy η induces a null-homotopy of the composite

$$L \longrightarrow X \overset{q_\sharp}{\simeq} DX \longrightarrow DL$$

and the pair (f, η) is called a Lagrangian if this sequence is a fibre sequence. A Poincaré form which admits a Lagrangian is called a metabolic form.

[20]It is a good exercise to show that this is so.

As before, with hyperbolic objects, we now define a Poincaré category whose Poincaré objects are precisely metabolic forms equipped with a Lagrangian. So let $(\mathcal{C}, Ϙ)$ be a Poincaré category. We define a hermitian structure[21] $Ϙ^{\mathrm{Met}}$ on $\mathrm{Arr}(\mathcal{C}) = \mathrm{Fun}(\Delta^1, \mathcal{C})$ by the formula

$$Ϙ^{\mathrm{Met}}(Y \to X) = \mathrm{fib}(Ϙ(X) \to Ϙ(Y))$$

and write $\mathrm{Met}(\mathcal{C}, Ϙ)$ for the hermitian category $(\mathrm{Arr}(\mathcal{C}), Ϙ^{\mathrm{Met}})$. We define a hermitian structure on the target functor $\mathrm{Arr}(\mathcal{C}) \to \mathcal{C}$ by the canonical map

$$Ϙ^{\mathrm{Met}}(Y \to X) = \mathrm{fib}(Ϙ(X) \to Ϙ(Y)) \longrightarrow Ϙ(X).$$

The following lemma is then merely spelling out definitions, I recommend the reader to work this out.

Lemma 1.60. *The hermitian category* $\mathrm{Met}(\mathcal{C}, Ϙ)$ *is Poincaré. The associated duality is given by*

$$D(Y \to X) = (\mathrm{fib}(DX \to DY) \to DY).$$

The target functor refines to a Poincaré functor $\mathrm{Met}(\mathcal{C}, Ϙ) \to (\mathcal{C}, Ϙ)$*. A Poincaré object for* $\mathrm{Met}(\mathcal{C}, Ϙ)$ *is a Lagrangian in the sense of Definition 1.59.*

Example 1.61. Similarly to Example 1.54 Lagrangians are also provided by geometry: Let M be a closed oriented and connected manifold of dimension d, and let W be a compact oriented $(d+1)$-dimensional manifold with $\partial W = M$ (as oriented manifold). Then the map $C^*(W) \to C^*(M)$ refines to a Poincaré object of $\mathrm{Met}(\mathrm{Perf}(\mathbb{Z}), \Omega^d Ϙ^{\mathrm{s}}_{\mathbb{Z}})$. To see this, we first have to provide a null-homotopy of the restriction of q_M. Ultimately, this comes from the following: Consider the cofibre sequence in $\mathrm{Perf}(\mathbb{Z})$

$$C_*(M) \longrightarrow C_*(W) \longrightarrow C_*(W, M)$$

Since the fundamental class of M is in the image of the boundary map $\Omega C_*(W, M) \to C_*(M)$, there is a preferred null-homotopy of the image of the fundamental class under the map $C_*(M) \to C_*(W)$. Then we need to consider the sequence

$$C^*(W) \longrightarrow C^*(M) \simeq \Omega^d C_*(M) \longrightarrow \Omega^d C_*(W)$$

and see whether it is a fibre sequence. The cofibre of the map $C^*(W) \to C^*(M)$ is given by $\Sigma C^*(W; M)$, and the definitions are made such that the induced map (after shifting once)

$$C^*(W, M) \longrightarrow \Omega^{d+1} C_*(W)$$

is given by cap product with the relative fundamental class of the manifold with boundary W, which Lefschetz duality shows to be an equivalence.

Finally, for later use, we will also introduce the notion of a cobordism, and a category whose Poincaré objects are cobordisms.

Definition 1.62. Let (X, q) and (X', q') be two Poincaré objects of a Poincaré category $(\mathcal{C}, Ϙ)$. A cobordism between (X, q) and (X', q') is a Lagrangian of $(X \oplus X', q \oplus -q')$.

Concretely, we can unravel the datum of a cobordism as follows: It consists of an object W equipped with maps $f \colon W \to X$ and $f' \colon W \to X'$ and a homotopy η from $f^*(q)$ to $f'^*(q')$ in

[21] Exercise: Show this is indeed a hermitian structure

the anima $\Omega^\infty Ϙ(W)$. This datum is required to satisfy the following non-degeneracy condition: The homotopy η gives rise to the commutative square

$$\begin{array}{ccc} W & \xrightarrow{q_\sharp \circ f} & DX \\ \big\downarrow{\scriptstyle q'_\sharp \circ f'} & & \big\downarrow{\scriptstyle Df} \\ DX' & \xrightarrow{Df'} & DW \end{array}$$

which is required to be a pullback. Notice that this condition specialises exactly to the earlier definition of a Lagrangian if $(X', q') = (0, 0)$.

We categorify this construction as follows: Consider the category $\mathrm{Fun}(\mathrm{TwArr}([1]), \mathcal{C})$ and the functor

$$Ϙ^{\mathrm{Cob}} \colon \mathrm{Fun}(\mathrm{TwArr}[1], \mathcal{C})^{\mathrm{op}} \simeq \mathrm{Fun}(\mathrm{TwArr}([1])^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}}) \to \mathrm{Fun}(\mathrm{TwArr}([1])^{\mathrm{op}}, \mathrm{Sp}) \xrightarrow{\lim} \mathrm{Sp}.$$

We recall here that for a category[22] K, the twisted arrow category $\mathrm{TwArr}(K)$ has as objects the morphisms of K, and a morphism from $\alpha \colon x \to y$ to a morphism $\alpha' \colon x' \to y'$ is a factorisation of α through α', i.e. a commutative square as the left square in the following diagram.

$$\begin{array}{ccccc} x & \xrightarrow{f} & x' & \xrightarrow{f'} & x'' \\ \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \alpha'} & & \big\downarrow{\scriptstyle \alpha''} \\ y & \xleftarrow[g]{} & y' & \xleftarrow[g']{} & y'' \end{array}$$

Composition is defined in the obvious way by glueing the above squares. Taking the source and target of a morphism then extends to a functor

$$\mathrm{TwArr}(K) \xrightarrow{(s,t)} K \times K^{\mathrm{op}}$$

which turns out to be the right fibration classified by the mapping anima functor of K.

Definition 1.63. We write $Q_1(\mathcal{C}, Ϙ)$ for the pair $(\mathrm{Fun}(\mathrm{TwArr}([1]), \mathcal{C}), Ϙ^{\mathrm{Cob}})$.

We recall that the category $\mathrm{TwArr}([1])$ looks like $\bullet \leftarrow \bullet \to \bullet$, so that an object of $Q_1(\mathcal{C}, Ϙ)$ consists of a span $X \leftarrow W \to X'$.

Lemma 1.64. *The functor $Ϙ^{\mathrm{Cob}}$ described above is a Poincaré structure, so $Q_1(\mathcal{C}, Ϙ)$ is a Poincaré category. The associated duality is given by sending a span $X \leftarrow W \to X'$ to the upper left span of the pullback diagram*

$$\begin{array}{ccc} \bar{W} & \longrightarrow & DX' \\ \big\downarrow & & \big\downarrow \\ DX & \longrightarrow & DW \end{array}$$

Exercise. Work out that the Poincaré objects of $Q_1(\mathcal{C}, Ϙ)$ are precisely cobordisms between Poincaré objects of $(\mathcal{C}, Ϙ)$.

[22] or an ∞-category, for that matter.

Exercise. Consider a diagram

$$\begin{tikzcd}
 & & W \times_{X'} W' \arrow[dl] \arrow[dr] & & \\
 & W \arrow[dl] \arrow[dr] & & W' \arrow[dl] \arrow[dr] & \\
X & & X' & & X''
\end{tikzcd}$$

Suppose that the bottom two spans are cobordisms. Then show that the big span is also a cobordism. We refer this big cobordism as the glued cobordism. Moreover, show that being cobordant induces an equivalence relation on the set of equivalences classes of Poincaré objects.

Definition 1.65. Let $(\mathcal{C}, \Omega)$ be a Poincaré category. We let $L_0(\mathcal{C}, \Omega)$ denote the set[23] of cobordism classes of Poincaré objects[24].

Exercise. Show that $L_0(\mathcal{C}, \Omega)$, under the operation of direct sum of Poincaré objects, is an abelian group.

Remark 1.66. We note that there are Poincaré functors $i\colon \mathrm{Met}(\mathcal{C}, \Omega) \to Q_1(\mathcal{C}, \Omega)$, given by sending a morphism $L \to X$ to the span $0 \leftarrow L \to X$, and the evaluation at the two endpoints give Poincaré functors $\mathrm{ev}_\ell, \mathrm{ev}_r\colon Q_1(\mathcal{C}, \Omega) \to (\mathcal{C}, \Omega)$. In fact, the following diagram of Poincaré functors will be relevant for us later:

$$\begin{tikzcd}
 & (\mathcal{C}, \Omega) \arrow[d, "\mathrm{const}"] \arrow[dr, equal] & \\
\mathrm{Met}(\mathcal{C}, \Omega) \arrow[r, "i"] \arrow[dr, "\mathrm{lag}"'] & Q_1(\mathcal{C}, \Omega) \arrow[r, "\mathrm{ev}_\ell"] \arrow[d, "p"] & (\mathcal{C}, \Omega) \\
 & \mathrm{Hyp}(\mathcal{C}) &
\end{tikzcd}$$

Here, p is the Poincaré functor adjoint[25] to the functor of stable categories given by sending a span $(X \leftarrow W \to X')$ to the fibre of the map $X \leftarrow W$. Consequently, the map $\mathrm{lag}\colon \mathrm{Met}(\mathcal{C}, \Omega) \to \mathrm{Hyp}(\mathcal{C})$ is adjoint to the functor extracting the Lagrangian L from the morphism $L \to X$.

1.6. **Adjunctions and the algebraic Thom isomorphism.** The main goal of the first part of this section is to show the following theorem.

Theorem 1.67. *Both of the forgetful functors*

$$\mathrm{Cat}_\infty^{\mathrm{p}} \longrightarrow \mathrm{Cat}_\infty^{\mathrm{h}} \longrightarrow \mathrm{Cat}_\infty^{\mathrm{ex}}$$

admit left and right adjoints. In particular, they all preserve all limits and colimits[26].

[23] Rather, it is canonically an abelian monoid under orthogonal direct sum.

[24] We might see later that this is indeed the 0th homotopy group of the L-spectrum of $(\mathcal{C}, \Omega)$.

[25] We will see next lecture that the construction Hyp is both left and right adjoint to the forgetful functor $\mathrm{Cat}_\infty^{\mathrm{p}} \to \mathrm{Cat}_\infty^{\mathrm{ex}}$; it is also quite easy to verify by hand.

[26] We discuss later that all of the involved categories are in fact complete and cocomplete.

For the functor $\mathrm{Cat}^{\mathrm{h}}_\infty \to \mathrm{Cat}^{\mathrm{ex}}_\infty$, this is considerably easier:

Lemma 1.68. *There is a functor* $\mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathrm{Cat}^{\mathrm{h}}_\infty$ *sending* $\mathcal{D}$ *to* $(\mathcal{D}, 0)$ *which is both left and right adjoint to the forgetful functor* $U\colon \mathrm{Cat}^{\mathrm{h}}_\infty \to \mathrm{Cat}^{\mathrm{ex}}_\infty$.

We'll first observe the following general lemma.

Lemma 1.69. *Let* $p\colon \mathcal{E} \to \mathcal{C}$ *be a cartesian fibration.*

(i) *If all fibres* $\mathcal{E}_c$ *have initial objects, then* p *has a left adjoint given by sending* c *to an initial object of* $\mathcal{E}_c$*, and*
(ii) *If all fibres* $\mathcal{E}_c$ *have terminal objects and the fibre functors* $\alpha^*\colon \mathcal{E}'_c \to \mathcal{E}_c$*, for morphisms* $\alpha\colon c \to c'$ *in* $\mathcal{C}$*, preserve terminal objects, then* p *has a right adjoint given by sending* c *to a terminal object of* $\mathcal{E}_c$.

Proof. In general the anima of maps in a Grothendieck construction, such as $\mathrm{Cat}^{\mathrm{h}}_\infty$ can be described as follows: Let $p\colon \mathcal{E} \to \mathcal{C}$ be a cartesian fibration, classified by a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{CAT}_\infty$. Let $e = (c, x \in F(c))$ and $e' = (c', x' \in F(c'))$ be two objects of $\mathcal{E}$. Then the fibre of the map

$$\mathrm{Map}_{\mathcal{E}}(e, e') \xrightarrow{p} \mathrm{Map}_{\mathcal{C}}(c, c')$$

over a point $\alpha\colon c \to c'$ is the anima $\mathrm{Map}_{F(c)}(x, \alpha^*(x'))$.

For the first part of the lemma, we need to show that above map is an equivalence, or equivalently that the fibre over every point of the base is contractible, provided $e = (c, x)$ is such that x is initial in $\mathcal{E}_c = F(c)$. But x being initial precisely says that the fibre, which is $\mathrm{Map}_{F(c)}(x, \alpha^*(x'))$ is contractible.

Likewise, if x' is terminal and all functors α^* preserve terminal objects, we again see that all fibres of the map under consideration are contractible. $\square$

Remark 1.70. The first part can also be shown by combining the following 2 facts:

(i) A general functor $p\colon \mathcal{E} \to \mathcal{C}$ admits a left adjoint if and only if for all objects c of $\mathcal{C}$, the slices $\mathcal{E}_{c/}$ have initial objects,
(ii) For a cartesian fibration $p\colon \mathcal{E} \to \mathcal{C}$, then canonical functors $\mathcal{E}_c \to \mathcal{E}_{c/}$ admit a right adjoint.

Indeed, from the second part, we deduce that the functor $\mathcal{E}_c \to \mathcal{E}_{c/}$ preserves colimits and hence the slices have initial objects if the fibres do.

Proof of Lemma 1.68. The fibres of the functor $\mathrm{Cat}^{\mathrm{h}}_\infty \to \mathrm{Cat}^{\mathrm{ex}}_\infty$ over a category $\mathcal{C}$ are, by definition given by $\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$, This is a stable category and in particular pointed, a zero object is the 0 functor. This is clearly preserved by precomposition, so the general lemma applies. $\square$

The construction of the adjoints of the forgetful functor $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{h}}_\infty$ is more involved – I will only sketch it here. Let $B\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ be a symmetric bilinear functor and consider the functor

$$\Omega^\infty B \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}))$$

to which one can associate an essentially unique bifibration $\mathrm{Pair}(\mathcal{C}, B) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$, i.e. a functor over the projection to $\mathcal{C}^{\mathrm{op}}$ which is a map of cocartesian fibrations such that its fibres are right fibrations

Remark 1.71. Consider in general a functor $\mathcal{C} \times \mathcal{D} \to \mathrm{An}$. Using unstraightening in one variable, we obtain a functor $\mathcal{D} \to \mathrm{RFib}(\mathcal{C}^{\mathrm{op}})$. The source functor gives a functor $\mathrm{RFib}(\mathcal{C}^{\mathrm{op}}) \to \mathrm{Cat}_\infty$, so we have a composition $\mathcal{D} \to \mathrm{RFib}(\mathcal{C}^{\mathrm{op}}) \to \mathrm{Cat}_\infty$ which unstraightens to a cocartesian fibration $\mathcal{E} \to \mathcal{D}$. The factorisation of the classifying functor through $\mathrm{RFib}(\mathcal{C}^{\mathrm{op}})$ provides a functor $\mathcal{E} \to \mathcal{C}^{\mathrm{op}}$ and by construction, the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr] \arrow[dr] & & \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[dl] \\ & \mathcal{D} & \end{tikzcd}$$

commutes and the horizontal map induces right fibrations on fibres. It is an explicit check that the horizontal map is a map of cocartesian fibrations. Thus presheaves on products can be unstraightened in four ways: to a cocartesian fibration, a cartesian fibration or the two possible choices of bifibrations (one over $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ and one over $\mathcal{C} \times \mathcal{D}^{\mathrm{op}}$)[27].

Informally, the category $\mathrm{Pair}(\mathcal{C}, B)$ consists of triples (X, Y, β) with $X \in \mathcal{C}$, $Y \in \mathcal{C}^{\mathrm{op}}$ and $\beta \in \Omega^\infty B(X, Y)$. A morphism from (X, Y, β) to (X', Y', β') then consists of maps $f\colon X \to X'$ and $g\colon Y' \to Y$ in $\mathcal{C}$ and an equivalence η between $f^*\beta$ and $g^*\beta'$ in $\Omega^\infty B(X', Y)$:

$$\begin{tikzcd} B(X,Y) \arrow[r, "g^*"] & B(X,Y') & B(X',Y') \arrow[l, "f^*"'] \\ \beta \arrow[r, mapsto] & g^*\beta \simeq f^*(\beta') & \beta' \arrow[l, mapsto] \end{tikzcd}$$

Now if B_{Ω} is the symmetric bilinear functor associated to a hermitian structure Ω on $\mathcal{C}$, we define a hermitian structure Ω_{Pair} on $\mathrm{Pair}(\mathcal{C}, \Omega) = \mathrm{Pair}(\mathcal{C}, B_{\Omega})$ by the pullback

$$\begin{tikzcd} \Omega_{\mathrm{Pair}}(X,Y,\beta) \arrow[r] \arrow[d] & \Omega(X) \arrow[d] \\ \mathrm{map}_{\mathcal{C}}(X,Y) \arrow[r] & B(X,X) \end{tikzcd}$$

where the lower horizontal map is induced by sending a morphism $f\colon X \to Y$ to the $f^*(\beta)$.

Proposition 1.72. *The hermitian structure Ω_{Pair} is Poincaré with associated duality given by $D(X, Y, \beta) = (X, Y, \tau_{X,Y}(\beta))$ where $\tau_{X,Y}\colon B(X,Y) \simeq B(Y,X)$ is the canonical equivalence furnished by the symmetry of B.*

Proof. See [CDH+20a, 7.3.2]. We write out the bilinear functor B_{Pair} here. It is given by the pullback

$$\begin{tikzcd} B_{\mathrm{Pair}}((X,Y,\beta),(X',Y',\beta')) \arrow[r] \arrow[d] & B_{\Omega}(X,X') \arrow[d, "{(\tau_{X,X'},\mathrm{id})}"] \\ \mathrm{map}_{\mathcal{C}}(X,Y') \oplus \mathrm{map}_{\mathcal{C}}(X',Y) \arrow[r] & B_{\Omega}(X',X) \oplus B_{\Omega}(X,X') \end{tikzcd}$$

From this, it is not hard to show that the Ω_{Pair} is perfect with duality as claimed. □

Proposition 1.73. *The association $(\mathcal{C}, \Omega) \mapsto \mathrm{Pair}(\mathcal{C}, \Omega)$ extends to a functor $\mathrm{Cat}^{\mathrm{h}}_\infty \to \mathrm{Cat}^{\mathrm{p}}_\infty$ which is right adjoint to the forgetful functor $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{h}}_\infty$. Likewise, the association $(\mathcal{C}, \Omega) \mapsto \mathrm{Pair}(\mathcal{C}, \Omega\Omega)$ extends to a functor $\mathrm{Cat}^{\mathrm{h}}_\infty \to \mathrm{Cat}^{\mathrm{p}}_\infty$ which is left adjoint to the forgetful functor $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{h}}_\infty$.*

[27] For further elaborations we refer to [CDH+20a, 7.1]

Proof. See [CDH+20a, 7.3.15 & 7.3.20]. ☐

Remark 1.74. The functor $\Omega\colon \mathcal{C} \to \mathcal{C}$ induces an equivalence between the hermitian categories $(\mathcal{C}, \Omega Ϙ)$ and $(\mathcal{C}, \Omega Ϙ(\Omega(-)))$. Sometimes it is more convenient to work with the latter instead of the former, which is why it is stated in this way in [CDH+20a, 7.3.20].

Let's work out some examples:

Example 1.75. Suppose $Ϙ = 0$. Then there is a canonical equivalence $\mathrm{Pair}(\mathcal{C}, \mathcal{C}^{\mathrm{op}}, 0) \simeq \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ and $Ϙ_{\mathrm{Pair}}(X, Y) \simeq \mathrm{map}_{\mathcal{C}}(X, Y)$. In other words $\mathrm{Pair}(\mathcal{C}, 0) \simeq \mathrm{Hyp}(\mathcal{C})$ where $\mathrm{Hyp}(\mathcal{C})$ is as described in Lemma 1.57.

Corollary 1.76. *The functor* $\mathrm{Hyp}\colon \mathrm{Cat}^{\mathrm{ex}}_{\infty} \to \mathrm{Cat}^{\mathrm{p}}_{\infty}$ *is left and right adjoint to the forgetful functor* $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Cat}^{\mathrm{ex}}_{\infty}$. *Consequently, we have an equivalence of functors* $\mathrm{Poinc} \circ \mathrm{Hyp} \simeq \mathrm{Core}$.

Proof. Only the second part requires an argument. Here we use that Poinc is corepresented by $(\mathrm{Sp}^{\omega}, Ϙ^{\mathrm{u}})$ on $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ and Core is corepresented by Sp^{ω} on $\mathrm{Cat}^{\mathrm{ex}}_{\infty}$[28]. ☐

For a Poincaré category $(\mathcal{C}, Ϙ)$, the counit of the adjunction exhibiting Hyp as left adjoint to the forgetful functor is a Poincaré functor $\mathrm{Hyp}(\mathcal{C}, Ϙ) \to (\mathcal{C}, Ϙ)$ which turns out to be the one described in Remark 1.58[29].

Example 1.77. Let $(\mathcal{C}, Ϙ)$ be a Poincaré category. Then the functor $B\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ is equivalent, via the duality in the second factor, to the functor $\mathrm{map}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$. The bifibration associated to the mapping space functor is the arrow category $\mathrm{Arr}(\mathcal{C}) = \mathrm{Fun}(\Delta^1, \mathcal{C})$, so we obtain a canonical equivalence $\mathrm{Pair}(\mathcal{C}, Ϙ) \simeq \mathrm{Arr}(\mathcal{C})$, concretely given by sending a triple $(X, Y, \beta \in B(X, Y))$ to the associated morphism $f_\beta\colon X \to DY$ under the equivalence $B(X, Y) \simeq \mathrm{map}(X, DY)$. Under this equivalence, a morphism $f\colon W \to Z$ therefore corresponds to the triple

$$(W, DZ, \beta_f \in B(X, DZ) \simeq \mathrm{map}_{\mathcal{C}}(X, D^2Z)),$$

and therefore the hermitian structure $Ϙ_{\mathrm{Pair}}$ on a morphism $f\colon W \to Z$ is given by the pullback

$$\begin{tikzcd} Ϙ(f) \arrow[r] \arrow[d] & Ϙ(W) \arrow[d] \\ \mathrm{map}(W, DZ) \arrow[r, "Df^*"] & \mathrm{map}(W, DW) \end{tikzcd}$$

Letting $L = \mathrm{fib}(W \to Z)$, it is a nice exercise to work out that $Ϙ(f) \simeq \mathrm{cofib}(Ϙ(W) \to Ϙ(L))$[30].

Thus, consider the auto equivalence of $\mathrm{Arr}(\mathcal{C})$ given by sending a morphism $f\colon L \to W$ to its cofibre $W \to Z$. Precomposition with this equivalence gives an equivalence $\mathrm{Arr}(\mathcal{C}) \simeq \mathrm{Pair}(\mathcal{C}, Ϙ)$ under which the Poincaré structure $Ϙ_{\mathrm{Pair}}$ corresponds to a Poincaré structure

$$\Sigma Ϙ_{\mathrm{Met}}(L \to X) = \Sigma \mathrm{fib}(Ϙ(X) \to Ϙ(L)).$$

In summary, for a Poincaré category $(\mathcal{C}, Ϙ)$ there is a canonical equivalence $\mathrm{Pair}(\mathcal{C}, Ϙ) \simeq \mathrm{Met}(\mathcal{C}, \Sigma Ϙ)$ where $\mathrm{Met}(\mathcal{C}, Ϙ)$ is as in Lemma 1.60.

[28] It is also instructive to prove the equivalence $\mathrm{Poinc}(\mathrm{Hyp}(\mathcal{C})) \simeq \mathrm{Core}(\mathcal{C})$ from the definitions.

[29] Conversely, given this map, one can just apply the mapping space criterion to see that Hyp is left adjoint to the forgetful functor, with this particular map as counit.

[30] See for instance [CDH+20a, 1.1.21] for a useful hint.

Remark 1.78. The counit of the adjunction exhibiting $\mathrm{Pair}(-,\Omega-)$ as left adjoint to the forgetful functor $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{h}}_\infty$ determines a Poincaré functor $\mathrm{Pair}(\mathcal{C},\Omega Ϙ) \longrightarrow (\mathcal{C},Ϙ)$. Under the above equivalence, this is the Poincaré functor $\mathrm{Met}(\mathcal{C},Ϙ) \longrightarrow (\mathcal{C},Ϙ)$ constructed in Lemma 1.60.

Remark 1.79. The Poincaré categories $\mathrm{Pair}(\mathcal{C},Ϙ)$ can be characterised as metabolic categories: We observe that there is an inclusion $\mathcal{C}^{\mathrm{op}} \to \mathrm{Pair}(\mathcal{C},Ϙ)$ given by $Y \mapsto (0,Y,0)$, and this functor admits a right adjoint given by the projection. In addition $Ϙ_{\mathrm{Pair}}$ vanishes when restricted to $\mathcal{C}^{\mathrm{op}}$, and the induced functor $\mathcal{C}^{\mathrm{op}} \to (\mathcal{C}^{\mathrm{op}})^\perp$ is an equivalence. Here, the orthogonal complement is meant with respect to the bilinear functor B_{Pair}, i.e. consists of all objects (X,Y,β) such that

$$B_{\mathrm{Pair}}((0,Y',0),(X,Y,\beta)) = 0.$$

This condition is equivalent to the following notion of a Lagrangian subcategory of a Poincaré category $(\mathcal{D},Ϙ)$. Let $\mathcal{L} \to \mathcal{D}$ be full stable subcategory. It is called isotropic if $Ϙ$ vanishes when restricted to $\mathcal{L}$, and it is called a Lagrangian if the induced map $\mathcal{L} \to \mathcal{L}^\perp$ is an equivalence and the inclusion $\mathcal{L} \to \mathcal{D}$ admits a right adjoint[31].

It turns out that these properties determine the pairings category: Given a Poincaré category $(\mathcal{D},Ϙ)$ with a Lagrangian $\mathcal{L}$, the inclusion $\mathcal{L} \to \mathcal{D}$ admits a right adjoint p. Let $q = \mathcal{D} \to \mathcal{D}^{\mathrm{op}} \to \mathcal{L}^{\mathrm{op}}$ be the composition of the duality on $\mathcal{D}$ and p. Consider the hermitian category $(\mathcal{L}^{\mathrm{op}}, q_!(Ϙ))$. Then there is a canonical equivalence $(\mathcal{D},Ϙ) \simeq \mathrm{Pair}(\mathcal{L}^{\mathrm{op}}, q_!(Ϙ))$.

Remark 1.80. We will see in Section 2.2 that $\mathrm{Cat}^{\mathrm{h}}_\infty$ carries a canonical symmetric monoidal structure which restricts to $\mathrm{Cat}^{\mathrm{p}}_\infty$, so that the functor $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{h}}_\infty$ is canonically symmetric monoidal, see Corollary 2.21. We deduce that it right adjoint $\mathrm{Pair}\colon \mathrm{Cat}^{\mathrm{h}}_\infty \to \mathrm{Cat}^{\mathrm{p}}_\infty$ is canonically lax symmetric monoidal.

As a consequence of the adjunctions above, we obtain what Ranicki called the algebraic Thom isomorphism:

Theorem 1.81. *Let $(\mathcal{C},Ϙ)$ be a Poincaré category. Then there is a canonical equivalence*

$$\mathrm{Poinc}(\mathrm{Met}(\mathcal{C},\Sigma Ϙ)) \xrightarrow{\simeq} \mathrm{Form}(\mathcal{C},Ϙ).$$

Proof. We have

$$\begin{aligned}\mathrm{Poinc}(\mathrm{Met}(\mathcal{C},\Sigma Ϙ)) &\simeq \mathrm{Map}_{\mathrm{Cat}^{\mathrm{p}}_\infty}((\mathrm{Sp}^\omega, Ϙ^{\mathrm{u}}), \mathrm{Met}(\mathcal{C},\Sigma Ϙ)) \\ &\simeq \mathrm{Map}_{\mathrm{Cat}^{\mathrm{p}}_\infty}((\mathrm{Sp}^\omega, Ϙ^{\mathrm{u}}), (\mathrm{Pair}(\mathcal{C},Ϙ))) \\ &\simeq \mathrm{Map}_{\mathrm{Cat}^{\mathrm{h}}_\infty}((\mathrm{Sp}^\omega, Ϙ^{\mathrm{u}}), (\mathcal{C},Ϙ)) \\ &\simeq \mathrm{Form}(\mathcal{C},Ϙ)\end{aligned}$$

□

Remark 1.82. We have indicated that the above equivalence has a preferred direction, so let us explain what the map is which the above (arguably abstract) argument shows to be an equivalence: It turns out that we need to associate to a Lagrangian $L \to X$ for $\Sigma Ϙ$ a canonical

[31] This can be reformulated in a way that looks closer in spirit to Lagrangians in the sense of ??, see [CDH+20b, 10.2.1 & 10.2.2]

form on the fibre of the map $L \to X$, now with respect to the Poincaré structure Ω. This is explicitly given as follows: We view the diagram

$$\begin{tikzcd} N \arrow[r] \arrow[d] & L \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$

as a morphism in $\mathrm{Met}(\mathcal{C}, \Sigma\Omega)$ from the left vertical to the right vertical map. Now, $L \to X$ being a Lagrangian specifies a Poincaré form $(q, \eta) \in \Sigma\Omega_{\mathrm{Met}}(L \to X)$, which can be pulled back to a form on $N \to 0$. Now we observe that

$$\Sigma\Omega_{\mathrm{Met}}(N \to 0) \simeq \Omega(N).$$

Later, we will give a concrete description of the inverse of the just described map, i.e. why a form on the fibre of $L \to X$ gives rise to a Poincaré structure on $L \to X$ with respect to the metabolic hermitian structure. This procedure is an instance of algebraic surgery.

Remark 1.83.

1.7. **Poincaré structures on module categories.** In this section we want to explain how Poincaré structures on $\mathrm{Perf}(R)$ for a ring spectrum R can be described in terms of module data. We recall from Corollary 1.31 that a Poincaré structure on a stable category $\mathcal{C}$ consists of a symmetric bilinear functor $B \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ and a linear functor $\Lambda \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ equipped with a map $\Lambda \Rightarrow (B \circ \Delta^*)^{tC_2}$.

For $\mathcal{C} = \mathrm{Perf}(R)$, these translate to *exact* functors $B \colon (\mathrm{Perf}(R) \otimes \mathrm{Perf}(R))^{\mathrm{op}} \to \mathrm{Sp}$ and $\Lambda \colon \mathrm{Perf}(R)^{\mathrm{op}} \to \mathrm{Sp}$. Here, we use the Lurie tensor product of stable categories [Lur17, Section 4.8][32]. It has the property that $\mathrm{Mod}(A) \otimes \mathrm{Mod}(B) \simeq \mathrm{Mod}(A \otimes B)$, or more precisely, the association $A \mapsto \mathrm{Mod}(A)$ refines to a symmetric monoidal functor $\mathrm{Alg}(\mathrm{Sp}) \to \mathrm{Pr}^{\mathrm{st}}$. We now recall the general Morita theory for colimit preserving functors between categories of modules, see [Lur17, 7.1.2.4]:

Proposition 1.84. *There is a canonical equivalence between* $\mathrm{Fun}^L(\mathrm{Mod}(R), \mathrm{Mod}(S))$ *and the category* $\mathrm{Mod}(R^{\mathrm{op}} \otimes S)$.

Proof. An $R^{\mathrm{op}} \otimes S$-module M can be used to obtain a colimit preserving functor $\mathrm{Mod}(R) \to \mathrm{Mod}(S)$ given by $- \otimes_R M$. Conversely, for a colimit preserving functor $F \colon \mathrm{Mod}(R) \to \mathrm{Mod}(S)$, we claim that $F(R)$ canonically refines to a $R^{\mathrm{op}} \otimes S$ module, through the functoriality of F. These constructions can be shown to refine to functors which are canonically inverse to each other[33]. □

Now recall that there is a functor $\mathrm{map}_R(-, R) \colon \mathrm{Mod}(R)^{\mathrm{op}} \to \mathrm{Mod}(R^{\mathrm{op}})$ which restricts to an equivalence $\mathrm{Perf}(R)^{\mathrm{op}} \simeq \mathrm{Perf}(R^{\mathrm{op}})$. We therefore need to construct exact functors $B \colon \mathrm{Perf}(R^{\mathrm{op}} \otimes R^{\mathrm{op}}) \to \mathrm{Sp}$ and $\Lambda \colon \mathrm{Perf}(R^{\mathrm{op}}) \to \mathrm{Sp}$. Equivalently, since $R^{\mathrm{op}} \otimes R^{\mathrm{op}} = (R \otimes R)^{\mathrm{op}}$, we need to construct colimit preserving functors

$$\mathrm{Mod}((R \otimes R)^{\mathrm{op}}) \xrightarrow{\tilde{B}} \mathrm{Sp} \xleftarrow{\tilde{\Lambda}} \mathrm{Mod}(R^{\mathrm{op}}).$$

By the above Morita theory, these are equivalently given by an $(R \otimes R)$-module M and an R-module N. Concretely, the functors B and Λ we care about are then given by

$$(X, Y) \mapsto \mathrm{map}_{R \otimes R}(X \otimes Y, M) \quad \text{and} \quad X \mapsto \mathrm{map}_R(X, N).$$

[32]There is a version for small categories and one for presentable ones.

[33]Making this precise is of course technically somewhat involved. I refer to .. for details.

We need to take care of two further structures: On the one hand, B is required to be a *symmetric* bilinear functor. It comes to no suprise that the equivalence

$$\mathrm{Fun}^{\mathrm{biex}}(\mathrm{Perf}(R)^{\mathrm{op}} \times \mathrm{Perf}(R)^{\mathrm{op}}, \mathrm{Sp}) \simeq \mathrm{Mod}(R \otimes R)$$

described above is C_2-equivariant for the two obvious flip actions.

Therefore, a symmetric refinement of the functor associated to B is equivalently given by lifting M to an object of $\mathrm{Mod}(R \otimes R)^{hC_2}$. Such an object is called a *module with involution over* R in [CDH+20a].

On the other hand, we need to translate the transformation $\Lambda \Rightarrow (B \circ \Delta^*)^{tC_2}$ to some datum involving M and N. The following lemma is needed for this:

Lemma 1.85. *Let M be a module with involution over R. Consider the associated linear functor* $\mathrm{Perf}(R)^{\mathrm{op}} \to \mathrm{Sp}$ *given by*

$$X \mapsto \mathrm{map}_{R\otimes R}(X \otimes X, M)^{tC_2}.$$

Under the equivalence of the category of such linear functors with $\mathrm{Mod}(R)$*, this functor corresponds to the object M^{tC_2} where the R-module structure is the canonical $(R \otimes R)^{tC_2}$-module structure restricted along the Tate diagonal $R \to (R \otimes R)^{tC_2}$.*

Proof. As indicated in the proof of Proposition 1.84, the R-module associated to such a linear functor is the value on R, with its R-module structure obtained by functoriality. In our case, this is therefore

$$\mathrm{map}_{R\otimes R}(R \otimes R, M)^{tC_2} \simeq M^{tC_2}$$

and one works out that the R-module structure is as claimed, see [CDH+20a, 3.2.4][34] □

So suppose now that Λ and B are the functors associated to an R-module N and a module with involution M over R. Again, by the Morita theory described above, a natural transformation $\Lambda \Rightarrow (B \circ \Delta^*)^{tC_2}$ then corresponds precisely to a map of R-modules $N \to M^{tC_2}$.

The collection of an R-module N, a module with involution M over R and an R-linear map $N \to M^{tC_2}$ is called a *module with genuine involution over* R in [CDH+20a, 3.2.2]. We arrive at the following (slightly informal) classification result:

Theorem 1.86. *Let R be a ring spectrum. Then a hermitian structure on* $\mathrm{Perf}(R)$ *is equivalently given by a module with genuine involution $(N, M, \alpha\colon N \to M^{tC_2})$ over R, and concretely the hermitian structure associated to this module with genuine involution is given by*

$$\begin{tikzcd}
Ϙ^{\alpha}_M(X) \arrow[r] \arrow[d] & \mathrm{map}_R(X, N) \arrow[d] \\
\mathrm{map}_{R\otimes R}(X \otimes X, M)^{hC_2} \arrow[r] & \mathrm{map}_R(X, M^{tC_2})
\end{tikzcd}$$

Clearly, one can upgrade this pointwise statement to an equivalence of categories. For this, we define the category of modules with genuine involution over R, denoted by $\mathrm{Mod}(NR)$[35],

[34] I am not super happy with what is written there. There must be a more transparent/formal argument.

[35] This is no accident: This category can be described as modules over the multiplicative norm of R see [CDH+20a, 3.2.3].

as the pullback

$$\begin{tikzcd} \mathrm{Mod}(NR) \arrow[r] \arrow[d] & \mathrm{Arr}(\mathrm{Mod}(R)) \arrow[d, "t"] \\ \mathrm{Mod}(R\otimes R)^{hC_2} \arrow[r] & \mathrm{Mod}(R) \end{tikzcd}$$

The upper horizontal functor sends a triple (N, M, α) to α, and the left vertical functor sends it to M. The right vertical functor is the target functor, and the lower horizontal functor sends M to M^{tC_2}, viewed as R-module via the Tate diagonal as explained earlier.

In view of the pullback description for hermitian structures obtained in Corollary 1.31, the Morita theory described above then induces a functor

$$\mathrm{Mod}(NR) \longrightarrow \mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathrm{Perf}(R)^{\mathrm{op}}, \mathrm{Sp})$$

which is an equivalence: It is the functor induced on pullbacks by functors which are equivalences by the above Morita theory.

Finally, we wish to analyse what it means for a hermitian structure on $\mathrm{Perf}(R)$, given by a module with genuine involution (N, M, α) to be a Poincaré structure.

For this we note that the equivalence

$$\mathrm{map}_{R\otimes R}(X \otimes Y, M) \simeq \mathrm{map}_R(X, \mathrm{map}_R(Y, M))$$

where, for sake of definiteness, the term $\mathrm{map}_R(Y, M)$ makes use of the R-module structure on M obtained throught the left unit $R = \mathbb{S}\otimes R \to R\otimes R$. The remaining R-module action on M then gives $\mathrm{map}_R(Y, M)$ an R-module structure which is used to form $\mathrm{map}_R(X, \mathrm{map}_R(Y, M))$. Note, however, that the C_2-action induces an equivalence between the the two R-module structures on M obtained through restriction along the left and right unit, respectively.

Therefore, the bilinear functor associated to M, is perfect in the sense of definition **??** if and only if the functor

$$\mathrm{map}_R(-, M)\colon \mathrm{Mod}(R)^{\mathrm{op}} \longrightarrow \mathrm{Mod}(R)$$

restricts to an equivalence

$$\mathrm{map}_R(-, M)\colon \mathrm{Perf}(R)^{\mathrm{op}} \longrightarrow \mathrm{Perf}(R).$$

To analyse when this is the case, we observe that M, viewed as an R-module (along either of the two unit maps) is itself perfect precisely says that the functor $\mathrm{map}_R(-, M)$ restricts to perfect modules. To obtain that it restricts to an equivalence, we know that this is the case if and only if applying this functor twice is canonically equivalent to the identity. Therefore, we arrive at the condition that the canonical map

$$R \longrightarrow \mathrm{map}_R(\mathrm{map}_R(R, M), M) \simeq \mathrm{map}_R(M, M)$$

is an equivalence. A module with involution M over R satisfying the two properties

- M is perfect as R-module, and
- the map $R \to \mathrm{map}_R(M, M)$ is an equivalence

therefore corresponds to the hermitian structure associated to (N, M, α) to be Poincaré. Such modules with involution are called *invertible* in [CDH+20a][36].

[36] It is worthwhile to contemplate the case where R is commutative, M is an R-module with R-linear C_2-action viewed as an $(R\otimes R)$ module via the multiplication map. In this case, M is an invertible module in our sense if and only if it is an invertible module in the usual sense, i.e. $\otimes$-invertible for the symmetric monoidal structure on $\mathrm{Mod}(R)$ furnished by the relative tensor product $\otimes_R$.

We end with some examples that come to mind (and will indeed be relevant later).

Example 1.87. Let R be a ring spectrum and M an invertible module with involution over R. It turns out that the category $\mathrm{Mod}(R)$ admits a t-structure whose coconnective part are the modules whose underlying spectrum is coconnective. In case R is connective, for instance if R is discrete, this is just the usual Postnikov t-structure. We can therefore consider the map

$$\mathrm{can}\colon \tau_{\geq n}(M^{tC_2}) \longrightarrow M^{tC_2}$$

with respect to this t-structure.

Example 1.88. Let R be a commutative ring, and view R as invertible module with (trivial) involution over R. Then the Tate-Frobenius map

$$R \longrightarrow (R \otimes R)^{tC_2} \longrightarrow R^{tC_2}$$

gives a module with genuine involution over R. The corresponding Poincaré structure will be denoted by $\mathfrak{P}^{\mathrm{t}}$, and for $R = \mathbb{S}$ we have $\mathfrak{P}^{\mathrm{t}}_{\mathbb{S}} = \mathfrak{P}^{\mathrm{u}}$, see Example 1.40.

1.8. **Animated Poincaré structures and form parameters.** In this section, we want to study certain Poincaré structures on $\mathrm{Perf}(R)$ where R is a discrete ring, relating them to form parameters in the sense of Bak or more generally Schlichting, see also [CDH+20a,].

In what follows, we shall always assume that the module with involution M over R is discrete, in which case being invertible implies that it is finitely generated projective (with respect to either of the two R-module structures).

Definition 1.89. Let $\mathcal{C}, \mathcal{D}$ be additive ∞-categories. A reduced functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *2-polynomial* if its cross effect B_F is additive in each variable separately. We denote by $\mathrm{Fun}^{\mathrm{red},2-\mathrm{poly}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the full subcategory of reduced 2-polynomial functors.

The category $\mathrm{Proj}(R)$ is an additive category, and the canonical functor $\mathrm{Proj}(R) \to \mathrm{Perf}(R)$, sending P to $P[0]$ is additive[37]. We then have the following theorem [CDH+20a, 4.2.15].

Theorem 1.90. *For any stable category $\mathcal{E}$, the functor $\mathrm{Proj}(R) \to \mathrm{Perf}(R)$ induces an equivalence*

$$\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathrm{Perf}(R)^{\mathrm{op}}, \mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{red},2-\mathrm{poly}}(\mathrm{Proj}(R)^{\mathrm{op}}, \mathcal{E}).$$

Proof sketch. Firstly, let $\mathcal{P}_\Sigma(\mathrm{Proj}(R))$ denote the free sifted cocompletion of $\mathrm{Proj}(R)$, i.e. the smallest full subcategory of $\mathcal{P}(\mathrm{Proj}(R)) = \mathrm{Fun}(\mathrm{Proj}(R)^{\mathrm{op}}, \mathrm{An})$ containing $\mathrm{Proj}(R)$ and which is closed under sifted colimits (i.e. filtered colimits and geometric realisations)[38] Then there is an equivalence $\mathcal{D}(R)_{\geq 0} \simeq \mathcal{P}_\Sigma(\mathrm{Proj}(R))$, and consequently an equivalence of categories of functors

$$\mathrm{Fun}^{\mathrm{sifted}}(\mathcal{D}(R)_{\geq 0}, \mathcal{E}) \simeq \mathrm{Fun}(\mathrm{Proj}(R), \mathcal{E})$$

provided $\mathcal{E}$ admits sifted colimits. One can get rid of this assumption by observing that $\mathcal{D}(R)_{\geq 0}$ is the generated under filtered colimits by $\mathrm{Perf}(R)_{\geq 0}$, so that there is a further equivalence

$$\mathrm{Fun}^{\mathrm{sifted}}(\mathcal{D}(R)_{\geq 0}, \mathcal{E}) \simeq \mathrm{Fun}^{f\Delta^{\mathrm{op}}}(\mathrm{Perf}(R)_{\geq 0}, \mathcal{E})$$

where the latter denotes functors which preserve socalled *finite geometric realisations* [CDH+20a, 4.2.28] i.e. the functor preserves colimits for simplicial diagrams which are left Kan extended

[37] In fact, it is the initial additive functor to a stable ∞-category.

[38] One can show that $\mathcal{P}_\Sigma(\mathrm{Proj}(R)) \simeq \mathrm{Fun}^{\pi}(\mathrm{Proj}(R)^{\mathrm{op}}, \mathrm{An})$ [Lur09a, Section 5.5.8].

along $\Delta^{\mathrm{op}}_{\leq n} \subseteq \Delta^{\mathrm{op}}$ for some n (note however, that the functor need *not* send a finite diagram to a finite diagram). Combining, we obtain an equivalence

$$\mathrm{Fun}^{f\Delta^{\mathrm{op}}}(\mathrm{Perf}(R)_{\geq 0}, \mathcal{E}) \simeq \mathrm{Fun}(\mathrm{Proj}(R), \mathcal{E})$$

which in fact holds for any stable ∞-category $\mathcal{E}$.

Under this equivalence, one shows that reduced 2-polynomial functors correspond precisely to reduced 2-excisive functors: First, one shows that 2-excisive functors preserve finite geometric realisations and obtains a functor from left to right which is fully faithful. One then needs to show essential surjectivity, see [CDH+20a, 4.2.34] for an argument. Playing around with op's, in particular replacing $\mathcal{E}$ by $\mathcal{E}^{\mathrm{op}}$, we arrive at an equivalence

$$\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathrm{Perf}(R)^{\mathrm{op}}_{\geq 0}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{red},2-\mathrm{poly}}(\mathrm{Proj}(R)^{\mathrm{op}}, \mathcal{E}).$$

Finally, the inclusion $\mathrm{Perf}(R)_{\geq 0} \subseteq \mathrm{Perf}(R)$ also induces an equivalence

$$\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathrm{Perf}(R)^{\mathrm{op}}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathrm{Perf}(R)^{\mathrm{op}}_{\geq 0}, \mathcal{E})$$

see [CDH+20a, 4.2.33]; this is an explicit argument analysing the adjunction obtained by right Kan extension along $\mathrm{Perf}(R)^{\mathrm{op}}_{\geq 0} \subseteq \mathrm{Perf}(R)^{\mathrm{op}}$ and the right adjoint of the inclusion of reduced 2-excisive functors in reduced functors, see however also the following remark. □

Remark 1.91. Given a reduced 2-excisive functor, $\Omega\!\!\!\!\!\;\text{?}$ —

Remark 1.91. Given a reduced 2-excisive functor, $\mathfrak{Q}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{E}$, one can write down a formula for $\mathfrak{Q}(\Omega X)$ in terms of $\mathfrak{Q}(X)$ and the cross effect of $\mathfrak{Q}$. Concretely, there is a fibre sequence

$$\mathfrak{Q}(\Omega X) \longrightarrow \Sigma^2 B_{\mathfrak{Q}}(X,X) \longrightarrow \Sigma^2\mathfrak{Q}(X).$$

I recommend that the reader works this fibre sequence out. This is a good informal reason for the equivalence

$$\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathrm{Perf}(R)^{\mathrm{op}}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathrm{Perf}(R)^{\mathrm{op}}_{\geq 0}, \mathcal{E})$$

as it shows how to inductively extend a reduced 2-excisive functor on $\mathrm{Perf}(R)^{\mathrm{op}}_{\geq 0}$ to $\mathrm{Perf}(R)^{\mathrm{op}}$ (because it is clear how to extend the cross effect).

Definition 1.92. By Theorem 1.90, any reduced 2-polynomial functor $Q\colon \mathrm{Proj}(R)^{\mathrm{op}} \to \mathcal{E}$ extends, in an essentially unique manner, to a reduced 2-excisive functor $\mathfrak{Q}\colon \mathrm{Perf}(R)^{\mathrm{op}} \to \mathcal{E}$. This extension is called the (non-abelian) *derived* functor of Q.

Remark 1.93.

Remark 1.94. There are notions of n-excisive and n-polynomial functors for all $n \geq 1$, and Theorem 1.90 is true in this generality. The case $n = 1$ is then the statement that any additive functor from $\mathrm{Proj}(R)^{\mathrm{op}}$ to a stable ∞-category extends uniquely to an exact functor $\mathrm{Perf}(R)^{\mathrm{op}}$ as indicated in the footnote.

Example 1.95. Let R be a ring and M a discrete and invertible module with involution over R. Consider the functors

$$Q^{\mathrm{gs}}_M, Q^{\mathrm{gq}}_M, Q^{\mathrm{ge}}_M\colon \mathrm{Proj}(R)^{\mathrm{op}} \longrightarrow \mathrm{Ab} \longrightarrow \mathrm{Sp}$$

given by

(i) $Q^{\mathrm{gs}}(P) = \mathrm{Hom}_{R\otimes_{\mathbb{Z}}R}(P \otimes_{\mathbb{Z}} P, M)^{C_2}$,
(ii) $Q^{\mathrm{gq}}(P) = \mathrm{Hom}_{R\otimes_{\mathbb{Z}}R}(P \otimes_{\mathbb{Z}} P, M)_{C_2}$, and
(iii) $Q^{\mathrm{ge}}(P) = \mathrm{Image}(\mathrm{Norm}\colon Q^{\mathrm{gq}}(P) \to Q^{\mathrm{gs}}(P))$.

These are the abelian groups of M-valued *symmetric, quadratic*, and *even* forms on P.

These functors are reduced 2-polynomial, and therefore give rise to hermitian structures on $\mathrm{Perf}(R)$ denoted by Ω^{gs}_M, Ω^{gq}_M, and Ω^{ge}_M[39].

Exercise. Let R be a ring and M a discrete and invertible module with involution over R. Consider the hermitian structures $\Omega^{\geq m}_M$ obtained from the modules with genuine involution $(\tau_{\geq m}M^{tC_2}, M, \mathrm{can}\colon \tau_{\geq m}M^{tC_2} \to M^{tC_2})$ of Example 1.87. Show that there are canonical equivalences

$$\Omega^{\mathrm{gs}}_M \simeq \Omega^{\geq 0}_M \quad , \quad \Omega^{\mathrm{ge}}_M \simeq \Omega^{\geq 1}_M \quad \text{and} \quad \Omega^{\mathrm{gq}}_M \simeq \Omega^{\geq 2}_M.$$

Show that for no other value of m, $\Omega^{\geq m}_M$ is a non-abelian derived hermitian structure. Deduce also that all of the above hermitian structures are Poincaré.

Remark 1.96. Schlichting showed that reduced 2-polynomial functors $Q\colon \mathrm{Proj}(R)^{\mathrm{op}} \to \mathrm{Ab}$ are equivalent to (generalised) form parameters by considering the maps

$$B_Q(R,R)_{C_2} \longrightarrow Q(R) \longrightarrow B_Q(R,R)^{C_2}$$

Therefore, every (generalised) form parameter (M, Q, α, β), and this includes the form parameters in the sense of Bak, gives rise to a hermitian structure on $\mathrm{Perf}(R)$ which is Poincaré if M is an invertible module with involution over R.

See [CDH+20a, 4.2.23] and the surrounding parts for details, further constructions, and examples.

Exercise. Show that the loop functor induces an equivalence of Poincaré categories

$$(\mathrm{Perf}(R), \Sigma^2 \Omega^{\geq m}_M) \simeq (\mathrm{Perf}(R), \Omega^{\geq m+1}_{-M}).$$

Here, $-M$ refers to the module with involution over R obtained from M by changing the C_2-action by a sign. Deduce the equivalences

$$(\mathrm{Perf}(R), \Sigma^2 \Omega^{\mathrm{gs}}_M) \simeq (\mathrm{Perf}(R), \Omega^{\mathrm{ge}}_{-M})$$

and

$$(\mathrm{Perf}(R), \Sigma^2 \Omega^{\mathrm{ge}}_M) \simeq (\mathrm{Perf}(R), \Omega^{\mathrm{gq}}_{-M}).$$

Exercise. Let R be a 2-torsion free commutative ring, viewed as a module with involution over R by letting C_2-act trivially. Show that

$$(\mathrm{Perf}(R), \Omega^{\mathrm{gq}}_R) \simeq (\mathrm{Perf}(R), \Omega^{\mathrm{ge}}_R) \quad \text{and} \quad (\mathrm{Perf}(R), \Omega^{\mathrm{ge}}_{-R}) \simeq (\mathrm{Perf}(R), \Omega^{\mathrm{gs}}_{-R}).$$

Remark 1.97. For a discrete ring R, the Poincaré category $(\mathrm{Perf}(R), \Omega^{\mathrm{gs}}_M)$ is the one we have to consider when trying to recover classical Grothendieck–Witt theory of R; *not* $(\mathrm{Perf}(R), \Omega^{\mathrm{s}}_M)$.

1.9. **Visible Poincaré structures and colimits in $\mathrm{Cat}^{\mathrm{p}}_\infty$.** First, we have the following general proposition, whose proof I only want to sketch.

Proposition 1.98. *The categories* $\mathrm{Cat}^{\mathrm{h}}_\infty$ *and* $\mathrm{Cat}^{\mathrm{p}}_\infty$ *are cocomplete and complete.*

Remark 1.99. From the fact that the functors $\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{h}}_\infty \to \mathrm{Cat}^{\mathrm{ex}}_\infty$ have all adjoints, it follows that each of the two functors preserves limits and colimits.

[39] The "g" stands for genuine.

Proof of Proposition 1.98. We take for granted here that $\mathrm{Cat}^{\mathrm{ex}}_\infty$ is complete and cocomplete. The case for $\mathrm{Cat}^{\mathrm{h}}_\infty$ is actually quite formal: Given a bicartesian fibration $\pi\colon \mathcal{E} \to \mathcal{C}$ over a bicomplete category $\mathcal{C}$ such that the fibres are also bicomplete, one finds that $\mathcal{E}$ is bicomplete. A concrete recipe for calculating a colimit of a functor $p\colon K \to \mathcal{E}$ is given as follows[40]: Consider a diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "\pi"] \\ K^{\triangleright} \arrow[r, "\bar{p}"] & \mathcal{C} \end{tikzcd}$$

such that $\bar{p}$ is a colimit cone with cone point $c \in \mathcal{C}$, viewed equivalently as a natural transformation $\Delta^1 \to \mathrm{Fun}(K, \mathcal{C})$ from πp to the constant functor at c. We may therefore reinterpret this as a commutative diagram

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathrm{Fun}(K, \mathcal{E}) \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed, "\tau"] & \mathrm{Fun}(K, \mathcal{C}) \end{tikzcd}$$

where a dotted arrow exists, since π_* is again cocartesian. One finds that for each $x \in K$, $\tau(-)(x)$ is a π-cocartesian morphism over the map $\pi(p(x)) \to c$, and that $\tau(1) \in \mathrm{Fun}(K, \mathcal{E})$ is a functor whose projection to $\mathcal{C}$ is constant at c, and therefore gives rise to a functor $\tau(1)\colon K \to \mathcal{E}_c$, of which one can take a colimit in $\mathcal{E}_c$. The claim is that then, the resulting object of $\mathcal{E}$ is a colimit of the original functor $p\colon K \to \mathcal{C}$[41].

Now to see that $\mathrm{Cat}^{\mathrm{p}}_\infty$ is also complete and cocomplete, one forms first the colimit or limit in $\mathrm{Cat}^{\mathrm{h}}_\infty$ and then shows that

(i) The colimit/limit in $\mathrm{Cat}^{\mathrm{h}}_\infty$ lies in fact in $\mathrm{Cat}^{\mathrm{p}}_\infty$, and
(ii) is indeed a colimit/limit there.

This is worked out in [CDH+20a, 6.1.4]. □

The following in [CDH+20a, 6.1.8].

Lemma 1.100. *The functors* $\mathrm{Poinc}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{An}$ *and* $\mathrm{Form}\colon \mathrm{Cat}^{\mathrm{h}}_\infty \to \mathrm{An}$ *preserve filtered colimits. Consequently* $(\mathrm{Sp}^\omega, \Omega^{\mathrm{u}})$ *is a compact object in* $\mathrm{Cat}^{\mathrm{p}}_\infty$ *and* $\mathrm{Cat}^{\mathrm{h}}_\infty$.

We will now be interested in particular diagrams in $\mathrm{Cat}^{\mathrm{p}}_\infty$. For a stable category $\mathcal{C}$, we recall that $\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ is stable, and hence tensored over Sp^ω. In other words, there is a functor

$$\mathrm{Sp}^\omega \times \mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \longrightarrow \mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$$

which restricts to a functor

$$\mathrm{Pic}(\mathbb{S}) \times \mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \longrightarrow \mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$$

having the property that for a Poincaré structure $\Omega\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$, the resulting functor

$$\mathrm{Pic}(\mathbb{S}) \longrightarrow \mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$$

[40] And the recipe for limits is similar

[41] For the construction, we have only used that π is a cocartesian fibration, and that $\mathcal{C}$ and the fibres of π have colimits. Exercise: Find out whether these conditions suffice, or what more is needed to conclude the argument.

lands again in the subspace of Poincaré structures on $\mathcal{C}$[42]. Hence, for a fixed Poincaré category $(\mathcal{C}, \Omega)$ we obtain a diagram

$$\begin{array}{ccccc} \mathrm{Pic}(\mathbb{S}) & \longrightarrow & \mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})^{\mathrm{poinc}} & \longrightarrow & \mathrm{Cat}^{\mathrm{p}}_{\infty} \\ & & \downarrow & & \downarrow \\ & & \{\mathcal{C}\} & \longrightarrow & \mathrm{Cat}^{\mathrm{ex}}_{\infty} \end{array}$$

and in fact, this construction is functorial in $(\mathcal{C}, \Omega)$, resulting in a functor

$$\mathrm{Pic}(\mathbb{S}) \times \mathrm{Cat}^{\mathrm{p}}_{\infty} \longrightarrow \mathrm{Cat}^{\mathrm{p}}_{\infty}$$

lying over the projection

$$\mathrm{Pic}(\mathbb{S}) \times \mathrm{Cat}^{\mathrm{ex}}_{\infty} \longrightarrow \mathrm{Cat}^{\mathrm{ex}}_{\infty}.$$

Now let X be an anima and let $\xi\colon X \to \mathrm{Pic}(\mathbb{S}) \subseteq \mathrm{Sp}^{\omega} \subseteq \mathrm{Sp}$ be a spherical fibration over X. From the above construction, ξ induces a functor

$$X \times \mathrm{Cat}^{\mathrm{p}}_{\infty} \longrightarrow \mathrm{Cat}^{\mathrm{p}}_{\infty}.$$

Definition 1.101. For an anima X with a spherical fibration $\xi\colon X \to \mathrm{Pic}(\mathbb{S})$, we let $((\mathrm{Sp}_X)^{\mathrm{f}}, \Omega^{\mathrm{v}}_{\xi})$ be the colimit of the functor $X \to \mathrm{Cat}^{\mathrm{p}}_{\infty}$ given by the above functor evaluated on $(\mathrm{Sp}^{\omega}, \Omega^{\mathrm{u}})$. It is called the visible Poincaré category associated to (X, ξ).

We'll add some small recollection on parametrised spectra here. So let X be an anima. We consider the category $\mathrm{Fun}(X, \mathrm{Sp})$ of X-parametrised spectra[43] We will denote the mapping spectrum in this category by $\mathrm{map}_X(-,-)$. The category $\mathrm{Fun}(X, \mathrm{Sp})$ is endowed with the pointwise symmetric monoidal structure denoted by $\otimes_X$ or simply $\otimes$ if X is understood. Given a map $f\colon X \to Y$, the restriction functor

$$f^*\colon \mathrm{Fun}(Y, \mathrm{Sp}) \to \mathrm{Fun}(X, \mathrm{Sp})$$

is canonically symmetric monoidal, and preserves limits and colimits. Therefore it has a right adjoint f_* and a left adjoint $f_!$, given by right and left Kan extension, respectively. In the extreme case where $Y = *$, this is simply taking a limit and a colimit over X, respectively. We define a candidate of an internal mapping object as follows: Given $\mathcal{F}, \mathcal{G}\colon X \to \mathrm{Sp}$, consider the composite

$$X \longrightarrow X \times X \simeq X^{\mathrm{op}} \times X \xrightarrow{\mathcal{F}^{\mathrm{op}}, \mathcal{G}} \mathrm{Sp}^{\mathrm{op}} \times \mathrm{Sp} \xrightarrow{\mathrm{map}} \mathrm{Sp}$$

which is the functor taking a point x to the spectrum of maps $\mathrm{map}_{\mathrm{Sp}}(\mathcal{F}_x, \mathcal{G}_x)$ with functoriality given by conjugation with an (invertible!) morphism in X. We denote this composite by $\hom_X(\mathcal{F}, \mathcal{G})$. Straight from the definitions, we obtain a canonical equivalence

$$\hom_X(\mathcal{F}, \hom_X(\mathcal{G}, \mathcal{H})) \simeq \hom_X(\mathcal{F} \otimes_X \mathcal{G}, \mathcal{H}).$$

In particular, by forming limits over X, we obtain an equivalence

$$r_* \hom_X(\mathcal{F}, \hom_X(\mathcal{G}, \mathcal{H})) \simeq r_* \hom_X(\mathcal{F} \otimes_X \mathcal{G}, \mathcal{H}).$$

[42] I.e. the collection of components of the maximal groupoid of $\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ on hermitian structures which are Poincaré, I'll write $\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})^{\mathrm{poinc}}$ for it.

[43] The category Sp could be replaced by any other bicomplete category for the discussion that is to come.

Now we observe that $r_* \hom_X \simeq \mathrm{map}_X$, showing that $\hom_X(\mathcal{F}, -)$ is right adjoint to $\mathcal{F} \otimes_X -$: For this, we recall that the spectrum of maps in a functor category to a stable category is calculated by an end, i.e. that we have a canonical equivalence

$$\mathrm{map}_X(\mathcal{F}, \mathcal{G}) \simeq \lim_{\mathrm{Tw}(X)} \mathrm{map}_{\mathrm{Sp}}(\mathcal{F}_x, \mathcal{G}_y).$$

Now we observe that the functor $\mathrm{Tw}(X) \to X \times X^{\mathrm{op}}$ is equivalent to the diagonal $X \to X \times X$ (since X is a groupoid). Therefore, the right hand side of the above equivalence is precisely the limit over X of the functor $\hom_X(\mathcal{F}, \mathcal{G})$ as claimed.

Again straight from the definitions, we see that for a map $f \colon X \to Y$ the functor f^* is compatible with the internal mapping object in the sense that

$$f^* \hom_Y(\mathcal{F}, \mathcal{G}) \simeq \hom_X(f^*\mathcal{F}, f^*\mathcal{G}).$$

Symmetric monoidal functors between closed symmetric monoidal categories which are also compatible with the internal mapping objects as just described are also called closed symmetric monoidal functors. It is a completely formal consequence that the left adjoint $f_!$ of a closed symmetric monoidal functor then satisfies the projection formula[44]

$$f_!(\mathcal{F}) \otimes_Y \mathcal{G} \simeq f_!(\mathcal{F} \otimes_X f^*\mathcal{G}).$$

Finally, there is an external product

$$- \boxtimes - \colon \mathrm{Fun}(X, \mathrm{Sp}) \times \mathrm{Fun}(Y, \mathrm{Sp}) \longrightarrow \mathrm{Fun}(X \times Y, \mathrm{Sp})$$

given by $(\mathcal{F} \boxtimes \mathcal{G})_{(x,y)} = \mathcal{F}_x \otimes \mathcal{G}_y$. Clearly, we have $\mathcal{F} \otimes_X \mathcal{G} = \Delta^*(\mathcal{F} \boxtimes \mathcal{G})$. In addition, for $\mathcal{F} \in \mathrm{Fun}(X, \mathrm{Sp})$, the functor $\mathcal{F} \boxtimes - \colon \mathrm{Fun}(Y, \mathrm{Sp}) \to \mathrm{Fun}(X \times Y, \mathrm{Sp})$ admits a right adjoint, denoted by $\hom^{\boxtimes}(\mathcal{F}, -) \colon \mathrm{Fun}(X \times Y, \mathrm{Sp}) \to \mathrm{Fun}(Y, \mathrm{Sp})$. Composing adjunctions, we get $\hom^{\boxtimes}(\mathcal{F}, \Delta_*(-)) \simeq \hom_X(\mathcal{F}, -)$. Concretely, for $\mathcal{H} \in \mathrm{Fun}(X \times Y, \mathrm{Sp})$ we have

$$\hom^{\boxtimes}(\mathcal{F}, \mathcal{H})_y = \mathrm{map}_X(\mathcal{F}_-, \mathcal{H}_{-,y}).$$

We come back to visible Poincaré categories. By Remark 1.99, we know that the underlying stable category of the visible Poincaré category associated to (X, ξ) is given by the colimit over the constant diagram indexed over X with value Sp^ω. This colimit can be calculated as follows:

Lemma 1.102. *There is a canonical equivalence between* $\mathrm{colim}_X \mathrm{Sp}^\omega$ *and the stable subcategory of* $\mathrm{Fun}(X, \mathrm{Sp})$ *generated by* $x_!(\mathbb{S})$ *where* x *ranges through* $\pi_0(X)$.

Proof. It is clear that the colimit is a subcategory of its idempotent completion, which can be calculated to be $\mathrm{colim}_X \mathrm{Sp}^\omega$, but now the colimit is taken in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ instead of $\mathrm{Cat}_\infty^{\mathrm{ex}}$. We first calculate this colimit by making use of the equivalence $\mathrm{Cat}_\infty^{\mathrm{perf}} \simeq \mathrm{Pr}^{\mathrm{st}}_\omega$. Therefore, we first have to calculate the colimit of the constant functor indexed on X with value Sp in the category $\mathrm{Pr}^{\mathrm{st}}_\omega$. The category $\mathrm{Pr}^{\mathrm{st}}_\omega$ is equivalent (by passing to the right adjoints on morphisms and doing nothing on the objects) to the opposite of the category of compactly generated presentable categories, with right adjoint functors which preserves filtered colimits[45]. The stable version of [Lur09a, 5.5.7.6] says that limits in this category are calculated in Cat_∞. It is then not hard to show that the limit in Cat_∞ over the constant diagram indexed over X with value

[44] Exercise: Prove this.

[45] And hence all small colimits

a category $\mathcal{C}$ is given by the category $\mathrm{Fun}(X, \mathcal{C})$[46]. We deduce that the colimit in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ is given by $\mathrm{Fun}(X, \mathrm{Sp})^\omega$. One then checks that the subcategory generated by $\{x_!(\mathbb{S})\}_{x\in\pi_0(X)}$ is indeed the colimit in $\mathrm{Cat}_\infty^{\mathrm{ex}}$. □

Lemma 1.103. *The visible Poincaré structure associated to an anima X equipped with a spherical fibration ξ is given by the pullback*

$$\begin{array}{ccc} Ϙ^{\mathrm{v}}_\xi(\mathcal{F}) & \longrightarrow & \mathrm{map}_X(\mathcal{F}, \xi) \\ \downarrow & & \downarrow \\ \mathrm{map}_{X\times X}(\mathcal{F}\boxtimes\mathcal{F}, \Delta_!(\xi)) & \longrightarrow & \mathrm{map}_X(\mathcal{F}, (\Delta^*\Delta_!(\xi))^{tC_2}) \end{array}$$

where the right vertical map is the composite

$$\xi \longrightarrow (\Delta^*\Delta_!\xi)^{hC_2} \longrightarrow (\Delta^*\Delta_!\xi)^{tC_2}$$

where we use that $\Delta_!(\xi) \in \mathrm{Fun}(X\times X, \mathrm{Sp})^{hC_2}$, and that the unit map of the adjunction is C_2-equivariant.

In particular, the duality of the visible Poincaré structure is given by $D_\xi(\mathcal{F}) = \hom^{\boxtimes}(\mathcal{F}, \Delta_!(\xi))$. There is an equivalence[47]

$$D_\xi(\mathcal{F}) = \hom^{\boxtimes}(\mathcal{F}, \Delta_!(\xi)) \simeq \hom^{\boxtimes}(\mathcal{F}, \Delta_!(\mathbb{S}_X)) \otimes \xi = D_{CW}(\mathcal{F}) \otimes \xi.$$

Here, we write D_{CW} for the duality $D_{\mathbb{S}_X}$ associated to the visible structure for the trivial spherical fibration $\mathbb{S}_X$ on X. This is classically called Costenoble-Waner duality, hence the symbol.

1.10. **Poincaré duality complexes.** Next, we aim to show that for certain anima X, including closed manifolds, we obtain a canonical Poincaré object in $((\mathrm{Sp}_X)^{\mathrm{f}}, Ϙ^{\mathrm{v}}_\nu)$, where ν is a particular spherical fibration which is the spherical fibration underlying the stable normal bundle in case X is a closed manifold.

For this we have to make the following definitions. Let X be a compact anima and let $r\colon X \to *$ and let $r_!$ and r_* be the left and right adjoints of $r^*\colon \mathrm{Sp} \to \mathrm{Fun}(X, \mathrm{Sp})$, i.e. $r_! = \mathrm{colim}_X$ and $r_* = \lim_X$ as described more generally above.

Since X is compact, it turns out that the functor r_* preserves all colimits, and dually that the functor $r_!$ preserves all limits. Such statements can be reduced to the case when X is finite, in which case it follows from the classical fact that finite limits commute with colimits in stable categories, and dually that finite colimits commute with limits.

The general Morita theory, adapted to parametrised spectra rather than module spectra[48] says that, the functor

$$\begin{array}{ccc} \mathrm{Fun}(X, \mathrm{Sp}) & \longrightarrow & \mathrm{Fun}^L(\mathrm{Fun}(X, \mathrm{Sp}), \mathrm{Sp}) \\ E & \longmapsto & r_!(- \otimes E) \end{array}$$

[46]Exercise: Use the straightening-unstraightening equivalence to describe limits/colimits in Cat_∞ and deduce this result.

[47]One can write down a map from right to left, and check that it induces an equivalence pointwise.

[48]For connected X, there is an equivalence $\mathrm{Fun}(X, \mathrm{Sp}) \simeq \mathrm{Mod}(\mathbb{S}[\Omega X])$, so the two situations ought to be similar.

is an equivalence of categories. In particular, one finds that there exists a unique object $D_X \in \mathrm{Fun}(X, \mathrm{Sp})$ having the property that there is an equivalence of functors

$$r_!(- \otimes D_X) \simeq r_*(-).$$

Now consider the constant functor $X \to \mathrm{Cat}_\infty^{\mathrm{p}}$ with value $(\mathrm{Sp}^\omega, \Omega^{\mathrm{u}})$. Its colimit is given by $((\mathrm{Sp}_{/X})^\omega, \Omega^{\mathrm{v}}_{r^*\mathbb{S}})$, and the underlying duality on $\mathrm{Fun}(X, \mathrm{Sp})^\omega$ is called Costenoble-Waner duality and denoted by D_{CW}. In addition, we find that for each point of X, the canonical functor $i_!\colon \mathrm{Sp}^\omega \to \mathrm{Fun}(X, \mathrm{Sp})^\omega$ is Poincaré and in particular preserves the dualities.

Lemma 1.104. *Let X be a compact anima, and let $E \in \mathrm{Fun}(X, \mathrm{Sp})^\omega$ and $F \in \mathrm{Fun}(X, \mathrm{Sp})$. Then there is a canonical equivalence $\mathrm{map}_X(E, F) \simeq r_!(D_{CW}(E) \otimes F)$.*

Proof. Since the category $\mathrm{Fun}(X, \mathrm{Sp})^\omega$ is generated by $i_!(\mathbb{S})$ under colimits, it suffices to prove the formula for $E = i_!(\mathbb{S})$. On the left hand side we obtain $i^*(F)$, and on the right hand side we note that $D_{CW}(i_!(\mathbb{S})) = i_!(D\mathbb{S}) = i_!(\mathbb{S})$ since $i_!$ is a Poincaré functor. Therefore, we have

$$r_!(D_{CW}(i_!(\mathbb{S})) \otimes F) = r_!(i_!(\mathbb{S}) \otimes F) = r_!i_!(\mathbb{S} \otimes i^*(F)) = i^*(F)$$

by the projection formula for $i_!$. □

With this at hand, we can also characterise the dualising spectrum as follows.

Lemma 1.105. *Let X be a compact anima. Then D_X is given by the Costenoble-Waner dual of $r^*(\mathbb{S})$ and corepresents the functor $r_!$.*

Proof. We have $r_*(E) = \mathrm{map}_X(r^*(\mathbb{S}), E) = r_!(D_{CW}(r^*(\mathbb{S})) \otimes E)$, so we have $D_X = D_{CW}(r^*(\mathbb{S}))$ as claimed. To see the second part, we then use that $D_{CW}^2 = \mathrm{id}$:

$$\mathrm{map}_X(D_X, E) = r_!(D_{CW}(r^*(\mathbb{S})) \otimes E) = r_!(r^*(\mathbb{S}) \otimes E) = r_!(E).$$

□

Definition 1.106. The parametrised spectrum D_X is called the dualising spectrum of X. A compact anima is called a Poincaré duality complex if the dualising spectrum D_X is a spherical fibration, i.e. factors through $\mathrm{Fun}(X, \mathrm{Pic}(\mathbb{S})) \subseteq \mathrm{Fun}(X, \mathrm{Sp})$. In this case, we call the dualising spectrum the Spivak normal fibration of X. If X is connected, we call the negative of the dimension of any fibre of D_X the formal dimension of X.

The unit of the adjunction (r^*, r_*) gives rise to a canonical map

$$c_X\colon \mathbb{S} \longrightarrow r_*(r^*(\mathbb{S})) \simeq r_!(D_X)$$

called the (Pontryagin-Thom) collapse map.

Conversely, any object $\mathcal{F} \in \mathrm{Fun}(X, \mathrm{Sp})$ equipped with a map $c\colon \mathbb{S} \to r_!(\mathcal{F})$ determines a canonical transformation

$$p_*(-) \Rightarrow p_!(- \otimes \mathcal{F}).$$

We say that c exhibits $\mathcal{F}$ as the dualising spectrum of X if this transformation is an equivalence.

Lemma 1.107. *Let X be a compact anima. Then there is a canonical equivalence*

$$D(X_+) \simeq r_!(D_X).$$

Proof. We consider the mapping spectrum $\mathrm{map}_X(r^*(\mathbb{S}), r^*(\mathbb{S}))$. Using the adjunction (r^*, r_*) and the defining property of dualising complexes, we obtain an equivalence

$$\mathrm{map}_X(r^*(\mathbb{S}), r^*(\mathbb{S})) \simeq \mathrm{map}_{\mathrm{Sp}}(\mathbb{S}, r_*(r^*\mathbb{S})) = r_!(D_X).$$

On the other hand, Using the $(r_!, r^*)$ adjunction, we obtain an equivalence

$$\mathrm{map}_X(r^*(\mathbb{S}), r^*(\mathbb{S})) \simeq \mathrm{map}_X(r_!r^*(\mathbb{S}), \mathbb{S}).$$

The lemma then follows from the equivalence $r_!r^*(\mathbb{S}) \simeq \Sigma^\infty_+ X$[49]. □

Remark 1.108. For a spherical fibration $\xi \in \mathrm{Fun}(X, \mathrm{Pic}(\mathbb{S}))$, the spectrum $r_!(\xi)$ is in fact a well-known object: It is equivalent to the Thom spectrum $\mathrm{M}\xi$ of ξ. Therefore, Lemma 1.107 says that the Spanier-Whitehead dual of a Poincaré duality complex is equivalent to the Thom spectrum of its Spivak normal fibration.

Next, we show that closed manifolds M are Poincaré duality complexes whose formal dimension coincides with the usual dimension of M. For this, we will view the stable normal bundle ν_M of M as a spherical fibration without changing the notation for it – sometimes this is written as $J(\nu_M)$ because the forgetful maps $\mathrm{BO} \times \mathbb{Z} \to \mathrm{BPL} \times \mathbb{Z} \to \mathrm{BTop} \times \mathbb{Z} \to \mathrm{Pic}(\mathbb{S})$ are all called J-homomorphisms. We have the following recognition principle for the Spivak normal fibration.

Proposition 1.109. *Let X be a compact anima equipped with an orientation local system $\mathcal{O}_X$ and a class $[X] \in H_n(X; \mathcal{O}_X)$ such that for any other local system $\mathcal{L}$ of abelian groups on X, the map*

$$- \cap [X] \colon H^k(X; \mathcal{L}) \longrightarrow H_{n-k}(X; \mathcal{L} \otimes \mathcal{O}_X)$$

is an isomorphism[50]*. Let $\mathcal{F}$ be a spherical fibration over X (of formal dimension $-n$) equipped with a map $c\colon \mathbb{S} \to r_!(\mathcal{F}) = \mathrm{M}\mathcal{F}$. Assume that its $\mathbb{Z}$-linearisation is equivalent to $\mathcal{O}_X[-n]$ such that under the Thom isomorphism*

$$H_0(\mathrm{M}\mathcal{F}; \mathbb{Z}) \cong H_n(X; \mathcal{O}_X)$$

the class $[c]$ is sent to the fundamental class $[X]$. Then $\mathcal{F}$ is the map $c\colon \mathbb{S} \to r_!(\mathcal{F})$ exhibits $\mathcal{F}$ as the Spivak normal fibration of X.

Proof. We need to show that for all X-parametrised spectra $\mathcal{G}\colon X \to \mathrm{Sp}$, the induced map

$$r_*(\mathcal{G}) \longrightarrow r_!(\mathcal{F} \otimes \mathcal{G})$$

is an equivalence. We observe that both functors commute with colimits and limits in $\mathcal{G}$; for the latter this is because $\mathcal{F} \otimes -$ is an equivalence since $\mathcal{F}$ is invertible. Moreover, since $\mathrm{Fun}(X, \mathrm{Sp})$ is generated by objects of the form $i_!(\mathbb{S})$ it suffices to check the equivalence for such objects. We note that in this case, both left and right hand side are bounded below spectra as they are given by a finite limit respectively colimits of pointwise bounded below objects. It therefore suffices to show that the map

$$r_*(\mathcal{G}) \otimes \mathbb{Z} \longrightarrow r_!(\mathcal{F} \otimes \mathcal{G}) \otimes \mathbb{Z}$$

[49] Exercise: Prove this.

[50] If X is in addition *finite* such anima are traditionally called Poincaré duality spaces of formal dimension n and closed manifolds are of course the standard examples.

is an equivalence in $\mathcal{D}(\mathbb{Z})$. We now use the commutative diagram (recall that r_* is essentially a finite limit)

$$\begin{tikzcd} \mathrm{Fun}(X,\mathrm{Sp}) \arrow[r, "r_*"] \arrow[d, "-\otimes\mathbb{Z}"] & \mathrm{Sp} \arrow[d, "-\otimes\mathbb{Z}"] \\ \mathrm{Fun}(X,\mathcal{D}(\mathbb{Z})) \arrow[r, "r_*"] & \mathcal{D}(\mathbb{Z}) \end{tikzcd}$$

And see that it therefore suffices to prove that the map

$$r_*(\mathcal{G}) \longrightarrow r_!((\mathcal{F}\otimes\mathbb{Z})\otimes\mathcal{G})$$

is an equivalence for all $\mathcal{G}\in\mathrm{Fun}(X,\mathcal{D}(\mathbb{Z}))$ and $r_*, r_!$ viewed as functors $\mathrm{Fun}(X,\mathcal{D}(\mathbb{Z}))\to\mathcal{D}(\mathbb{Z})$. Again, both of these functors commute with limits and colimits, so by the (pointwise) Whitehead and Postnikov towers of $\mathcal{G}$, it suffices to show that the map

$$r_*(\mathcal{L}) \longrightarrow r_!(\mathcal{O}_X[-n]\otimes\mathcal{L})$$

is an equivalence for all local systems of abelian groups $\mathcal{L}$ on X. But $r_*(\mathcal{L}) = C^*(X;\mathcal{L})$ and $r_!(\mathcal{O}_X[-n]\otimes\mathcal{L}) = C_*(X;\mathcal{O}_X\otimes\mathcal{L})[-n]$, and the induced map between them is given by cap product with the class corresponding to $[c]$ under the Thom isomorphism

$$H_0(\mathrm{M}\mathcal{F};\mathbb{Z}) \cong H_0(\mathrm{M}\mathcal{O}_X;\mathbb{Z}) \cong H_n(X;\mathcal{O}_X).$$

By assumption, this class is $[X]$, so the map is an equivalence by assumption. □

Corollary 1.110. *Let M be a a closed manifold. Then the geometric Pontryagin-Thom collapse map $c_M\colon \mathbb{S}\to \mathrm{M}\nu_M$ exhibits the stable normal bundle ν_M of M as the dualising spectrum of M.*

Proof. The orientation local system of M is determined by $w_1(TM) = w_1(\nu_M)$. Therefore, the $\mathbb{Z}$-linearisation of ν_M is equivalent to the orientation local system of M. Furthermore, the Pontryagin–Thom collapse map has geometric degree 1. Therefore, under the Thom isomorphism

$$H_0(\mathrm{M}\nu;\mathbb{Z}) \cong H_n(M;\mathcal{O}_M)$$

the class $[c_M]$ corresponds to a generator, which is a fundamental class for $[M]$. Thus, classical Poincaré duality for M shows that M is a Poincaré duality complex and that the underlying spherical fibration of ν_M is the Spivak normal fibration on M. □

Remark 1.111. We also deduce the following: Consider a geometric degree one normal map, i.e. a pullback diagram

$$\begin{tikzcd} \nu_M \arrow[r, "\bar{f}"] \arrow[d] & E \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

where f has degree 1[51]. Then the composite

$$\mathbb{S} \longrightarrow \mathrm{M}\nu_M \longrightarrow \mathrm{M}E$$

[51]defined appropriately if we do not assume M and N to be oriented.

exhibits the underlying spherical fibration of E as the Spivak normal fibration of N so that $J(E) \simeq J(\nu_N)$. To see this, we consider the commutative diagram

$$\begin{tikzcd} H_0(\mathrm{M}\nu_M;\mathbb{Z}) \arrow[r] \arrow[d] & H_0(\mathrm{M}E;\mathbb{Z}) \arrow[d] \\ H_n(M;\mathcal{O}_M) \arrow[r] & H_n(N;\mathcal{O}_n) \end{tikzcd}$$

whose vertical maps are isomorphisms and where the lower horizontal map sends $[M]$ to $[N]$ by the degree 1 assumption.

Remark 1.112. Suppose that X is a Poincaré duality space in the sense of Proposition 1.109. Then it is also a Poincaré duality complex in our sense: Using Proposition 1.109 we only need to find a spherical fibration $\mathcal{F}$ over X, equipped with a map $\mathbb{S} \to \mathrm{M}\mathcal{F}$ giving rise to the fundamental class of $[X]$ under the Thom isomorphism.

This is precisely the classical existence statement for Spivak normal fibrations of Poincaré duality spaces due to Spivak. He gives a concrete construction of it as follows: By assumption, X is homotopy equivalent to a finite CW-complex, and then also to a finite simplicial complex. Any such complex can be embedded into $\mathbb{R}^n$ for suitably large n. One can then thicken up the embedding to a closed codimension zero embedding and therefore obtain a manifold N with boundary (if X is a manifold, think of this as being a closed tubular neighbourhood of the embedded manifold; a space isomorphic to the disk bundle of the normal bundle). This thickening can be chosen to retract onto to given embedding of X. In particular, one can restrict this retraction to the boundary ∂N of N, and obtain a map $\partial N \to X$. The theorem of Spivak is then that the (homotopy) fibres of this map are (have the homotopy type of) spheres if and only if X satisfies Poincaré duality in the sense of Proposition 1.109. It is instructive to work this out for simply connected X, where one does not have to worry about local systems.

Remark 1.113. This shows that there is a bundle theoretic obstruction for a Poincaré duality space X to be homotopy equivalent to a closed manifold: Its Spivak normal fibration must admit a lift to a stable vector bundle (in the appropriate version, depending on whether X is to be equivalent to a smooth, piecewise-linear, or topological manifold).

This obstruction does not always vanish. In addition, if this obstruction vanishes, each reduction of the Spivak normal fibration to a vector bundle gives rise to a candidate of a closed manifold which might be homotopy equivalent to X, more concretely, one obtains a socalled degree one normal map from a manifold M to X. Associated to this degree one normal map is the surgery obstruction; an invariant in the quadratic L-theory of the group ring of the fundamental group of X (or maybe more canonically in the quadratic L-theory of the visible Poincaré structure on $((\mathrm{Sp}_{/X})^\omega, \Omega^{\mathrm{v}}_{D_X})$. If the dimension of M is at least 5, and this obstruction vanishes, then M is h-cobordant to a homotopy equivalence $M' \to X$.

One can also classify how many manifolds are homotopy equivalent to X in a similar way. Here, there are two versions: One calculates the set of h-cobordism classes of homotopy equivalences to X, and one calculates the set of s-cobordism classes of simple homotopy equivalences to X. For the latter (which is the appropriate notion if one wants to classify manifolds up to homeomorphism or diffeomorphism by the s-cobordism theorem) one needs a more refined version of the surgery obstruction, using an L-theory which also takes care of Whitehead torsions of the maps $q_\sharp\colon Z \to DZ$ of Poincaré objects (Z,q). While such an

L-spectrum can be defined by some means, as of now, we do not know (and I do not expect) that this can be described simply as the L-theory of a suitable Poincaré category.

Exercise. Let M be closed manifold of dimension n. Assume (or take for granted that this can always be arranged) that the anima associated to M can be written as the cofibre of a map $\kappa\colon S^{n-1} \to M_0$ where M_0 is the anima associated to an $(n-1)$-dimensional CW complex. Show that the map κ is stably null homotopic if and only if the spherical fibration underlying the normal bundle of M is trivial. If this happens, one says that the top cell of M stably splits off.

Exercise. Classify the closed manifolds M whose top cell splits off (unstably, not stably).[52] Gather examples of manifolds whose top cell splits off stably.

Exercise. Let M be a closed oriented manifold of dimension n whose top cell stably splits off. Show that M satisfies Poincaré duality over the sphere spectrum, i.e. that

$$\mathrm{map}(\mathbb{S}^n, \Sigma^\infty_+ M) \simeq \mathrm{map}(\Sigma^\infty_+ M, \mathbb{S}).$$

Here is another nice exercise about the obstruction to finding a vector bundle reduction of the Spivak normal fibration of a Poincaré duality complex in general (it makes use of several things, so don't worry if you can't figure it out immediately).

We note that Proposition 1.109 shows that a simply connected finite anima X is a Poincaré duality complex if there exists a class $[X] \in H_n(X;\mathbb{Z})$ such that cap product with $[X]$ induces an isomorphism

$$H^k(X;\mathbb{Z}) \cong H_{n-k}(X;\mathbb{Z}).$$

Exercise. Now we consider the anima $X = \mathrm{cofib}(S^4 \xrightarrow{f} (S^2 \vee S^3))$ where $f = [\iota_2, \iota_3] + \eta^2 \in \pi_4(S^2 \vee S^3)$. Here, $[\iota_2, \iota_3]$ is the standard Whitehead product, i.e. the attaching map of the 5-cell for the standard CW structure on $S^2 \times S^3$, and η^2 refers to the composite

$$S^4 \xrightarrow{\Sigma\eta} S^3 \xrightarrow{\eta} S^2 \longrightarrow S^2 \vee S^3$$

in which η is the classical Hopf map and the last map is the canonical inclusion.

The Exercise is now to show that X is a Poincaré duality complex, but not homotopy equivalent to a closed (say smooth) manifold.

Remark 1.114. It turns out that dimension 5 is the first dimension to construct easy examples of this kind. In fact, in dimensions ≤ 3, every Poincaré duality complex admits a vector bundle reduction. In dimension 4, every *orientable* Poincaré duality complex admits a vector bundle reduction, but there exists non-orientable Poincaré duality complexes of formal dimension 4 whose Spivak normal fibration does not admit a vector bundle reduction. This is due to Hambleton and myself and is most likely what one would call a folklore result.

Warning 1.115. The paper I have on the arXiv about the reducibility of Poincaré duality complexes contains an error in the proof of Lemma 3.2, and the lemma is in fact wrong. I have a different argument, using L-theoretic methods, so the claim about the vanishing of the one differential is certainly correct (as also follows from Hambleton's argument). I do believe that there is an a priori argument for the vanishing of the needed differential, but as of now don't have one. I do intend to update this paper, and apologise for not having done so so far. There is no excuse for it. The rest of the paper might be a nice read nevertheless, and also

[52] Here, I've used the fun fact that cyclic groups are not acyclic (in the sense of homology).

contains a proof of the above exercise. A more direct proof analyses spin vector bundles on X, which is the approach I recommend to take when solving the exercise.

To finish this section, we come back to the visible Poincaré structure associated to a Poincaré duality complex X.

Proposition 1.116. *The visible Poincaré structure $Ϙ^{\mathrm{v}}_{D_X}$, when evaluated on D_X itself, is given by the following pullback*

$$\begin{tikzcd} Ϙ^{\mathrm{v}}_{D_X}(D_X) \arrow[r] \arrow[d] & r_!(D_X) \arrow[d] \\ r_!(D_X)^{hC_2} \arrow[r] & r_!(D_X)^{tC_2} \end{tikzcd}$$

where the C_2-action on $r_!(D_X)$ is trivial. In particular, the canonical map $c_X \colon \mathbb{S} \to r_!(D_X)$ determines a canonical form on D_X which turns out to be Poincaré.

Proof. We recall from Lemma 1.103 that the linear part of the visible Poincaré structure is given by $\mathrm{map}_X(-, D_X)$. Evaluated on D_X, and using Lemma 1.105, we get the formula for the linear part. For the bilinear part, one obtains the following: Let us denote the Costenoble-Waner duality on $\mathrm{Fun}(X, \mathrm{Sp})^\omega$ by D_{CW}. We recall that this is the duality one obtains on the colimit of the constant functor on X with values $(\mathrm{Sp}^\omega, Ϙ^{\mathrm{u}})$. Since it is a duality, it suffices to provide a formula for $B_{D_X}(-,-)$ on objects of the form $D_{CW}(L)$. In this case, we obtain the following formula, see [CDH+20a, 4.4.10]:

$$B_{D_X}(D_{CW}(L), D_{CW}(L')) \simeq \operatorname*{colim}_{x \in X} \left[(L_x \otimes L'_x) \otimes (D_X)_x\right].$$

The symmetries on both sides are the obvious ones. Since $D_X = D_{CW}(r^*(\mathbb{S}))$, we obtain

$$B_{D_X}(D_X, D_X) \simeq \operatorname*{colim}_{x \in X} \left[\mathbb{S} \otimes \mathbb{S} \otimes (D_X)_x\right]$$

with trivial C_2-action on the D_X part. This shows the formula for the value of $Ϙ^{\mathrm{v}}_{D_X}$ on D_X.

Alternatively, one can also use that for PD complexes X and Y, we find that $D_X \boxtimes D_Y$ is the dualising spectrum of $X \times Y$, or more precisely that the map

$$\mathbb{S} \simeq \mathbb{S} \otimes \mathbb{S} \xrightarrow{c_X \otimes c_Y} r_!(D_X) \otimes s_!(D_Y) = (r \times s)_!(D_X \boxtimes D_Y)$$

exhibits $D_X \boxtimes D_Y$ as the dualising spectrum for $X \times Y$; Here $r \colon X \to *$ and $s \colon Y \to *$ are the canonical maps to the terminal anima. In particular, we find that

$$\mathrm{map}_{X \times X}(D_X \boxtimes D_X, \Delta_!(\mathcal{F})) \simeq r_!(\mathcal{F})$$

for any X-parametrised spectrum $\mathcal{F}$, since maps out of $D_X \boxtimes D_X$ corepresents the functor $(r \times r)_!$, see Lemma 1.105.

The Pontryagin–Thom collapse map $\mathbb{S} \to r_!(D_X)$ therefore determines a map of $(X \times X)$-parametrised spectra $D_X \boxtimes D_X \to \Delta_!(D_X)$, which by adjunction gives a map $D_X \to \hom^{\boxtimes}(D_X, \Delta_!(D_X))$, and we need to see that this map is an equivalence to obtain a Poincaré form as claimed. This requires mainly not getting confused[53]. □

Definition 1.117. Let X be a Poincaré duality complex. The just described Poincaré object (D_X, q^{v}_X) for $((\mathrm{Sp}_{/X})^{\mathrm{f}}, Ϙ^{\mathrm{v}}_{D_X})$ is called the visible signature class of X.

[53] I'll try to add details later.

2. Grothendieck–Witt and L-spectra

2.1. **(Split) Poincaré-Verdier sequences and additive functors.** We first recall the basic properties of Verdier quotients of stable categories. In general, a functor $p\colon \mathcal{C} \to \mathcal{D}$ determines a full subcategory of $\mathcal{C}$, namely the kernel of p, i.e. all objects x of $\mathcal{C}$ such that $p(x)$ is a zero object in $\mathcal{D}$. For a Verdier projection, this subcategory completely recovers the functor p up to equivalence: We have a diagram

$$\begin{tikzcd} \ker(p) \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ 0 \arrow[r] & \mathcal{D} \end{tikzcd}$$

which is always a pullback. Then p is a Verdier projection if and only of the diagram is a pushout, i.e. the canonical map $\mathcal{C}/\ker(p) \to \mathcal{D}$ is an equivalence. $\mathcal{D}$ is then referred to as the Verdier quotient of $\mathcal{C}$ by $\ker(p)$, and a sequence as above (i.e. a bifibre sequence in $\mathrm{Cat}^{\mathrm{ex}}_\infty$) is called a Verdier sequence.

It turns out that one can define $\mathcal{D}$ as the Dwyer-Kan localisation of $\mathcal{C}$ at the collection W of morphisms whose fibre lie in $\mathcal{C}_0 = \ker(p)$; it then so happens that $\mathcal{C}[W^{-1}]$ also has the correct universal property in the world of stable ∞-categories, that is:

(i) The localisation $\mathcal{C}[W^{-1}]$ is stable,
(ii) The functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ is exact,
(iii) Given an exact functor $\mathcal{C} \to \mathcal{E}$ which inverts the morphisms of W (or equivalently sends $\mathcal{C}_0$ to zero objects), the induced functor $\mathcal{C}[W^{-1}] \to \mathcal{E}$ is exact as well.

In particular, this proves that $\mathrm{Cat}^{\mathrm{ex}}_\infty$ has cofibres. We have used this fact already, of course, but the above gives an explicit construction of cofibres.

We note that given a Verdier projection $p\colon \mathcal{C} \to \mathcal{D}$ we may pass to Ind-categories and obtain a map $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ which has a right adjoint (essentially given by precomposition with p under the equivalence $\mathrm{Ind}(-) \simeq \mathrm{Fun}^{\mathrm{ex}}((-)^{\mathrm{op}}, \mathrm{Sp})$). This right adjoint is fully faithful since $\mathcal{C} \to \mathcal{D}$ is a DK localisation. Therefore we find that the projection $\mathcal{C} \to \mathcal{D}$ always admits a fully faithful ind right adjoint $\mathcal{D} \to \mathrm{Ind}(\mathcal{C})$.

A Verdier sequence $\mathcal{C}_0 \to \mathcal{C} \to \mathcal{D}$ is called *split* if the projection (and hence also the inclusion) has both left and right adjoints (which are then, as discussed earlier, automatically fully faithful). Phrased for the right adjoint, the condition simply is that the ind right adjoint $\mathcal{D} \to \mathrm{Ind}(\mathcal{C})$ takes image in the full subcategory $\mathcal{C} \subseteq \mathrm{Ind}(\mathcal{C})$. In this case the projection is called a split Verdier projection (and the inclusion a split Verdier inclusion).

Example 2.1. Let $\mathcal{C}$ be a stable category. Then the target functor $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ is a split Verdier projection. There are not many more examples: Every split Verdier projection is a pullback of this one, see [CDH+20b, A.2.11].

Here are other names for split Verdier sequences.

Example 2.2. A split Verdier sequence $\mathcal{E} \to \mathcal{C} \to \mathcal{D}$ is equivalent to giving a semi-orthogonal decomposition of $\mathcal{C}$ into $\mathcal{E}$ and $\mathcal{D}$; that it to give two full stable subcategories, such that $\mathrm{map}(\mathcal{E}, \mathcal{D})$ is trivial and such that for each object X of $\mathcal{C}$, there is a fibre sequence

$$X_{\mathcal{E}} \longrightarrow X \longrightarrow X_{\mathcal{D}}$$

with $X_{\mathcal{E}} \in \mathcal{E}$ and $X_{\mathcal{D}}$ in $\mathcal{D}$. In other words, a semi-orthogonal decomposition is in turn the same thing as giving a t-structure on $\mathcal{C}$ where the connective and coconnective parts are *stable* subcategories (here, $\mathcal{E}$ is the connective part and $\mathcal{D}$ is the coconnective part).

This datum is also called a stable recollement of $\mathcal{C}$; see [CDH+20b, Appendix A.1 & A.2] for a thorough discussion of (split) Verdier sequences.

Definition 2.3. A Poincaré functor $(p, \eta)\colon (\mathcal{C}, Ϙ) \to (\mathcal{D}, \bar{Ϙ})$ is a Poincaré-Verdier projection[54] if the underlying functor is a Verdier projection and $\bar{Ϙ}$ is the left Kan extension of $Ϙ$ along $p^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$[55].

A PV projection is called split if the underlying functor p is a split Verdier projection.

Remark 2.4. As indicated earlier, it turns out that the left Kan extension of a hermitian structure along an exact functor is again a hermitian structure, thereby giving the association $\mathcal{C} \mapsto \mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ a covariant functoriality making the cartesian fibration $\mathrm{Cat}_\infty^{\mathrm{h}} \to \mathrm{Cat}_\infty^{\mathrm{ex}}$ also cocartesian. See [CDH+20a, 1.4] for details.

Remark 2.5. A Poincaré functor $p\colon (\mathcal{C}, Ϙ) \to (\mathcal{D}, Ϙ')$ which admits a fully faithful left adjoint $g\colon \mathcal{D} \to \mathcal{C}$ such that $Ϙ' \simeq (g^{\mathrm{op}})^* Ϙ$ is a split PV projection. To see this, we first recall that an exact functor p with a fully faithful left adjoint is a right Bousfield localisation, and hence a Verdier projection. The condition that $Ϙ' \simeq (g^{\mathrm{op}})^* Ϙ$ says that $Ϙ'$ is the left Kan extension of $Ϙ$ along p^{op}, as g^{op} is the right adjoint of p^{op}. Finally, we need to argue that p also admits a right adjoint, namely $D \circ gD'$.

$$\mathrm{map}_{\mathcal{C}}(X, DgD'Y) \simeq \mathrm{map}_{\mathcal{C}}(gD'Y, DX) \simeq \mathrm{map}_{\mathcal{D}}(D'Y, pDX) \simeq \mathrm{map}_{\mathcal{D}}(D'Y, D'pX) \simeq \mathrm{map}_{\mathcal{D}}(pX, Y)$$

as needed. See [CDH+20b, 1.1] for many more details and characterisations of PV projections and PV inclusions.

Remark 2.6. As a word of warning: The left adjoint of the projection p in a split PV sequence need *not* be itself a Poincaré functor (although it is canonically hermitian[56]). In particular, a split PV sequence is not splitted in $\mathrm{Cat}_\infty^{\mathrm{p}}$, contrary to the case of split Verdier sequences in $\mathrm{Cat}_\infty^{\mathrm{ex}}$.

Example 2.7. Let $(\mathcal{C}, Ϙ)$ be a Poincaré category. Then the canonical functor $\mathrm{Met}(\mathcal{C}, Ϙ) \to (\mathcal{C}, Ϙ)$ from Lemma 1.60 is a split PV projection. And again, there are not many more: Every split PV projection is a pullback of this one, see [CDH+20b, Theorem 8.2.9]

Definition 2.8. A diagram of Poincaré categories

$$\begin{tikzcd} (\mathcal{C}, Ϙ) \arrow[r] \arrow[d] & (\mathcal{C}', Ϙ') \arrow[d, "p"] \\ (\mathcal{D}, \bar{Ϙ}) \arrow[r] & (\mathcal{D}', \bar{Ϙ}') \end{tikzcd}$$

is called a Poincaré-Verdier square[57] if it is a pullback in $\mathrm{Cat}_\infty^{\mathrm{p}}$ and the vertical maps are PV projections. Likewise it is called a split PV square is in addition the vertical maps are split PV projections. A (split) PV square as above with $(\mathcal{D}, \bar{Ϙ}) = 0$ is called a (split) PV sequence.

[54] PV projection for short

[55] or rather that the canonical map $p_! Ϙ \to Ϙ'$ adjoint to the hermitian structure η on p is an equivalence

[56] Exercise: Convince yourself of this.

[57] PV square for short

Exercise. A PV square is a pushout in $\mathrm{Cat}^{\mathrm{p}}_\infty$; [CDH+20b, 8.5.2 (iii)]

Example 2.9. The vertical and horizontal sequences appearing Remark 1.66 are split PV sequences. For the horizontal one, we can apply the argument of Remark 2.5. For the second one, writing down the left adjoint of p is a bit tedious, however, there is a general criterion for when a map to a hyperbolic category is a split PV projection which applies here, see e.g. [CDH+20b, 10.1.2 & 8.4.1].

Definition 2.10. Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{An}$ be a reduced functor. We say that $\mathcal{F}$ is additive if it sends split PV squares to pullback squares. We denote by $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{An})$ the full subcategory of $\mathrm{Fun}^{\mathrm{red}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{An})$ on additive functors.

Remark 2.11. An additive functor preserves products so the forgetful map $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{An}) \to \mathrm{An}$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{An})) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{An}).$$

Since Poinc is corepresentable, it preserves limits. The same is true for Core. We therefore obtain:

Proposition 2.12. *The functor* $\mathrm{Poinc}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{An}$ *is additive. Likewise, the composite* $\mathrm{Core}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathrm{An}$ *is additive.*

Recall that $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An}) \subseteq \mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{An})$ is the full subcategory on the group-like objects, i.e. those $\mathbb{E}_\infty$-anima for which the shear map $X \oplus X \to X \oplus X$ is an equivalence. From the diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An})) \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{An})) \arrow[d] \\
\mathrm{Fun}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An})) \arrow[r] & \mathrm{Fun}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{An}))
\end{tikzcd}$$

and the fact the the vertical functors and the lower horizontal functor are fully faithful, we deduce that the upper horizontal functor is fully faithful as well.

Definition 2.13. An additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{An}$ is called group-like if it lies in the subcategory

$$\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Grp}_{\mathbb{E}_\infty}) \subseteq \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{An}).$$

Example 2.14. Neither Poinc nor Core are group-like additive functors. It is one of the basic properties of (connective) K-theory that the functor

$$\mathrm{Cat}^{\mathrm{ex}}_\infty \xrightarrow{K} \mathrm{An}$$

is group-like and additive. In fact, it is the initial such functor equipped with a map from Core. The next goal we aim for is to define a functor $\mathrm{GW}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{An}$, equipped with a transformation from Poinc, and show that it is the initial group-like additive functor under Poinc. Hence, informally, GW is the analog of connective algebraic K-theory for Poincaré categories.

The main goal for the next three sections is to indicate a proof of the following theorem.

Theorem 2.15. *The inclusion*

$$\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An})) \subseteq \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_\infty, \mathrm{An})$$

has a left adjoint.

In fact, we will explicitly construct this left adjoint. To the best of my knowledge, this is a not a consequence of some abstract adjoint functor theorem (in particular, we remark here that both the categories $\mathrm{Cat}_\infty^{\mathrm{p}}$ and An are compactly generated presentable categories, so the category of functors between them is huge). But in any case, it will be crucial for our purposes to construct this adjoint, in order to prove some basic properties about it.

Remark 2.16. The inclusion

$$\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{ex}}, \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An})) \subseteq \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{ex}}, \mathrm{An})$$

also has a left adjoint, where additive functors are defined similarly as we did by discarding all Poincaré structures everywhere. The proof we give in [CDH+20b] also gives this statement, which was of course well-known before. K-theory is then, as indicated above, the image of the Core functor under this left adjoint.

Exercise. Let $\mathcal{F}\colon \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathrm{An}$ be a group-like additive functor. Show that the map

$$\mathcal{F}(\mathrm{Met}(\mathcal{C}, Ϙ)) \longrightarrow \mathcal{F}(\mathrm{Hyp}(\mathcal{C}))$$

induced by $\mathrm{lag}\colon \mathrm{Met}(\mathcal{C}, Ϙ) \to \mathrm{Hyp}(\mathcal{C})$ is an equivalence. Show that the same is not true for a general additive functor which is not group-like.

Remark 2.17. Consider the functor $\mathcal{F} = \mathrm{Poinc}$. In this case, using the algebraic Thom isomorphism we have to investigate the map

$$\mathrm{Form}(\mathcal{C}, \Omega Ϙ) \longrightarrow \mathrm{Core}(\mathcal{C}).$$

The fibres of this map over a point X are given by $\Omega^\infty Ϙ(X)$, which is of course not trivial in general. In particular, the conclusion of the previous exercise fails for non group-like additive functors in general.

To motivate the technical parts that come now, let's recall very briefly what the K-theory space of a stable category is. Namely, we define $K(\mathcal{C}) = \Omega|\mathrm{Span}(\mathcal{C})|$. Here, $\mathrm{Span}(\mathcal{C})$ is the span category of $\mathcal{C}$ whose objects are the same as the objects of $\mathcal{C}$ and where morphisms from X to Y are spans

$$X \leftarrow Z \rightarrow Y$$

and composition is obtained from glueing spans.

Of course, one has to make this precise (we will do so below). In analogy, we will define $\mathrm{GW}(\mathcal{C}, Ϙ)$ as $\Omega|\mathrm{Cob}(\mathcal{C}, Ϙ)|$ where $\mathrm{Cob}(\mathcal{C}, Ϙ)$ is the Cobordism category of $(\mathcal{C}, Ϙ)$: objects are the Poincaré objects in $\mathcal{C}$, and morphisms between them are cobordisms of Poincaré objects. Composition is again induced by glueing cobordisms.

It turns out that making such construction precise is most convenient in terms of Segal anima which we will recall below. However, to obtain the appropriate Segal anima from a Poincaré category, we will use more technical tools: monoidal structures and cotensors. This is the topic of the next section.

2.2. **Symmetric monoidal structures, internal mapping objects, and cotensors.** For the construction of the left adjoint claimed in Theorem 2.15, in fact also at several other places, we will make use of a (co)tensoring of $\mathrm{Cat}_\infty^{\mathrm{h}}$ over Cat_∞. It is crucial that this construction is functorial in all input data. The plan in this section is to explain the statements one needs in order to arrive that pointwise constructions such as cotensors (which we will give below) and the "internal" functor categories indicated in Remark 1.49 can be promoted to functors. Later we will also discuss tensors. This section will contain mainly no proofs.

The basic ingredient making everything work is a symmetric monoidal structure on $\mathrm{Cat}_\infty^{\mathrm{h}}$. We recall that $\mathrm{Cat}_\infty^{\mathrm{ex}}$ is equipped with a symmetric monoidal structure which we denoted by $(\mathcal{C},\mathcal{C}')\mapsto \mathcal{C}\otimes\mathcal{C}'$ [Lur17, 4.8] which we take for granted in this course.

Construction 2.18. Let $(\mathcal{C},\Omega)$ and $(\mathcal{C},\Omega')$ be hermitian categories. We define a functor $\Omega\otimes\Omega'\colon \mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}'^{\mathrm{op}}\to \mathrm{Sp}$ as follows: There is a canonical functor $\beta\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C}'^{\mathrm{op}}\to\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}'^{\mathrm{op}}$. We can then form

$$\beta_!(\Omega\boxtimes\Omega')\in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}'^{\mathrm{op}},\mathrm{Sp})$$

This functor is reduced by inspection. We can the consider its image under the left adjoint of the inclusion

$$\mathrm{Fun}^{\mathrm{red},2-\mathrm{exc}}(\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}'^{\mathrm{op}},\mathrm{Sp})\subseteq \mathrm{Fun}^{\mathrm{red}}(\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}'^{\mathrm{op}},\mathrm{Sp})$$

and define it to be $\Omega\otimes\Omega'$. Consequently, we obtain a hermitian category $(\mathcal{C}\otimes\mathcal{C}',\Omega\otimes\Omega')$.

For the proof of the following proposition, see [CDH+20a, 5.1.3].

Proposition 2.19. *The linear part of $\Omega\otimes\Omega$ is given by $\Lambda_\Omega\otimes\Lambda_{\Omega'}$, and the bilinear part is given by $B_\Omega\otimes B_{\Omega'}$.*

Exercise. Show that $(\mathcal{C},\Omega)\otimes(\mathrm{Sp}^\omega,\Omega^{\mathrm{u}})\simeq(\mathcal{C},\Omega)\simeq(\mathrm{Sp}^\omega,\Omega^{\mathrm{u}})\otimes(\mathcal{C},\Omega)$ and that $(\mathcal{C},\Omega)\otimes(\mathcal{C},\Omega')$ is Poincaré if both $(\mathcal{C},\Omega)$ and $(\mathcal{C}',\Omega')$ are Poincaré, with duality given by $D\otimes D'\colon \mathcal{C}^{\mathrm{op}}\otimes\mathcal{C}'^{\mathrm{op}}\to\mathcal{C}\otimes\mathcal{C}'$.

One then has the following, [CDH+20a, 5.2.7].

Theorem 2.20. *There exists a symmetric monoidal structure on $\mathrm{Cat}_\infty^{\mathrm{h}}$ whose tensor bifunctor sends $((\mathcal{C},\Omega),(\mathcal{C}',\Omega'))$ to the hermitian category $(\mathcal{C},\Omega)\otimes(\mathcal{C}',\Omega')$ of Construction 2.18. Its tensor unit is $(\mathrm{Sp}^\omega,\Omega^{\mathrm{u}})$. The forgetful functor $\mathrm{Cat}_\infty^{\mathrm{h}}\to\mathrm{Cat}_\infty^{\mathrm{ex}}$ refines to a symmetric monoidal functor.*

I recommend to prove the following corollary as an exercise[58].

Corollary 2.21. *The symmetric monoidal structure restricts to $\mathrm{Cat}_\infty^{\mathrm{p}}$. In particular, the functor $\mathrm{Cat}_\infty^{\mathrm{p}}\to\mathrm{Cat}_\infty^{\mathrm{h}}$ is symmetric monoidal.*

Recall the internal functor categories of Remark 1.49. As expected, they indeed satisfy the universal property of internal mapping objects, see [CDH+20a, 6.2.7]: I'll change notation now and write $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\Omega),(\mathcal{C}',\Omega'))$ for the hermitian category $(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C},\mathcal{C}'),\mathrm{nat}^{\Omega}_{\Omega'})$.

Proposition 2.22. *There is a canonical equivalence*

$$\mathrm{Map}_{\mathrm{Cat}_\infty^{\mathrm{h}}}((\mathcal{C},\Omega)\otimes(\mathcal{C}',\Omega'),(\mathcal{C}'',\Omega''))\simeq \mathrm{Map}_{\mathrm{Cat}_\infty^{\mathrm{h}}}((\mathcal{C},\Omega),(\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}',\Omega'),(\mathcal{C}'',\Omega'')))).$$

Again, the following corollary is a formal argument in ∞-categories which I recommend work out[59].

Corollary 2.23. *The formation of functor categories extends to a functor*

$$(\mathrm{Cat}_\infty^{\mathrm{h}})^{\mathrm{op}}\times\mathrm{Cat}_\infty^{\mathrm{h}}\longrightarrow\mathrm{Cat}_\infty^{\mathrm{h}}.$$

In addition, for fixed $(\mathcal{C},\Omega)$ the functor $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\Omega),-)$ is a right adjoint of the functor $(\mathcal{C},\Omega)\otimes -$. Consequently, the symmetric monoidal structure on $\mathrm{Cat}_\infty^{\mathrm{h}}$ is closed, *so that the tensor product of $\mathrm{Cat}_\infty^{\mathrm{h}}$ preserves colimits in each variable.*

---

[58] For instance, prove in general that a subcategory of a symmetric monoidal category which contains the tenor unit and is closed under tensor products is itself a symmetric monoidal category and that the inclusion refines to a symmetric monoidal functor.

[59] Hint: Use the Yoneda lemma.

In fact, one can upgrade the equivalence of Proposition 2.22 to the internal functor categories:

Corollary 2.24. *There is a canonical equivalence*

$$\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\Omega)\otimes(\mathcal{C}',\Omega'),(\mathcal{C}'',\Omega''))\simeq\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\Omega),\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}',\Omega'),(\mathcal{C}'',\Omega''))).$$

If all categories involved are Poincaré, then we may pass to Poincaré objects of the equivalence of Corollary 2.24. We then obtain an equivalence

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{p}}_\infty}((\mathcal{C},\Omega)\otimes(\mathcal{C}',\Omega'),(\mathcal{C}'',\Omega''))\simeq\mathrm{Map}_{\mathrm{Cat}^{\mathrm{p}}_\infty}((\mathcal{C},\Omega),(\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}',\Omega'),(\mathcal{C}'',\Omega'')))),$$

and as before can therefore conclude:

Corollary 2.25. *The formation of functor categories extends to a functor*

$$(\mathrm{Cat}^{\mathrm{p}}_\infty)^{\mathrm{op}}\times\mathrm{Cat}^{\mathrm{p}}_\infty\longrightarrow\mathrm{Cat}^{\mathrm{p}}_\infty.$$

For fixed Poincaré category $(\mathcal{C},\Omega)$*, the functor* $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\Omega),-)$ *is a right adjoint of the functor* $(\mathcal{C},\Omega)\otimes-$*. Consequently, the symmetric monoidal structure on* $\mathrm{Cat}^{\mathrm{p}}_\infty$ *is* closed*, so that the tensor product of* $\mathrm{Cat}^{\mathrm{p}}_\infty$ *preserves colimits in each variable as well.*

Remark 2.26. By the explicit description of the internal mapping objects, we find that the functors $\mathrm{Cat}^{\mathrm{p}}_\infty\to\mathrm{Cat}^{\mathrm{h}}_\infty\to\mathrm{Cat}^{\mathrm{ex}}_\infty$ are not only symmetric monoidal (i.e. preserve the tensor product) but also *closed* (i.e. preserve internal mapping objects). Consequently, the left adjoint of these functors satisfy a projection formula as we've discussed in an earlier exercise: For a closed symmetric monoidal functor $f^*\colon\mathcal{C}\to\mathcal{D}$ between closed symmetric monoidal categories, with left adjoint $f_!\colon\mathcal{D}\to\mathcal{C}$, the projection formula is the canonical equivalence

$$f_!(X)\otimes Y\simeq f_!(X\otimes f^*(Y)).$$

This implies, for instance, that hyperbolic categories form an ideal in $\mathrm{Cat}^{\mathrm{p}}_\infty$, something we will use later when discussing noncommutative Poincaré–Witt motives.

Next, we come to cotensors. Let $\mathcal{I}$ be a category and $(\mathcal{C},\Omega)$ a hermitian category. Consider the composite

$$\Omega^{\mathcal{I}}\colon\mathrm{Fun}(\mathcal{I},\mathcal{C})^{\mathrm{op}}\simeq\mathrm{Fun}(\mathcal{I}^{\mathrm{op}},\mathcal{C}^{\mathrm{op}})\xrightarrow{\Omega}\mathrm{Fun}(\mathcal{I}^{\mathrm{op}},\mathrm{Sp})\xrightarrow{\lim_{\mathcal{I}}}\mathrm{Sp}.$$

The following is [CDH+20a, 6.3.2].

Lemma 2.27. *The functor* $\Omega^{\mathcal{I}}$ *is a hermitian structure on* $\mathrm{Fun}(\mathcal{I},\mathcal{C})$*. Its associated bilinear functor is given by the formula*

$$B^{\mathcal{I}}(F,G)=\lim_{i\in\mathcal{I}^{\mathrm{op}}}B_{\Omega}(F(i),G(i)).$$

If $\mathcal{I}$ *is a finite category, then the linear part of* $\Omega^{\mathcal{I}}$ *is given by*

$$\Lambda^{\mathcal{I}}(F)=\lim_{i\in\mathcal{I}^{\mathrm{op}}}\Lambda_{\Omega}(F(i)).$$

If Ω *is non-degenerate (i.e. the bilinear functor is described by a pre-duality* D*), and* $\mathcal{C}$ *admits* $(\mathcal{I}_{i/})^{\mathrm{op}}$*-shaped limits*[60]*, then* $\Omega^{\mathcal{I}}$ *is also non-degenerate with pre-duality* $D^{\mathcal{I}}$ *given by*

$$(D^{\mathcal{I}}(F))(i)=\lim_{j\in(\mathcal{I}_{i/})^{\mathrm{op}}}D(F(j)).$$

[60] For instance, if the slices are themselves finite categories.

Definition 2.28. We define the cotensor $(\mathcal{C}, \Omega)^{\mathcal{I}}$ of $(\mathcal{C}, \Omega)$ with $\mathcal{I}$ to be the hermitian category $(\mathcal{C}^{\mathcal{I}} = \mathrm{Fun}(\mathcal{I}, \mathcal{C}), \Omega^{\mathcal{I}})$.

Example 2.29. The category $Q_1(\mathcal{C}, \Omega)$ defined in Definition 1.63 is the cotensor of $(\mathcal{C}, \Omega)$ with $\mathrm{TwArr}([1])$. It is no accident that this is Poincaré, as the following proposition shows.

Proposition 2.30. *Let $\mathcal{J}_K$ be the face poset of a finite simplicial complex K. Then the cotensoring construction $(\mathcal{C}, \Omega) \mapsto (\mathcal{C}, \Omega)^{\mathcal{J}_K}$ preserves Poincaré categories. If $f\colon K \to L$ is a simplicial map of finite simplicial complexes, and $\mathcal{J}_K \to \mathcal{J}_L$ is the induced functor of face posets, then the hermitian functor $(\mathcal{C}, \Omega)^{\mathcal{J}_L} \to (\mathcal{C}, \Omega)^{\mathcal{J}_K}$ is Poincaré.*

Proof. [CDH+20a, 6.6.1 & 6.6.2]. □

Remark 2.31. There is a general criterion which implies that the hermitian functor $(\mathcal{C}, \Omega)^{\mathcal{I}} \to (\mathcal{C}, \Omega)^{\mathcal{J}}$ associated to a functor $\mathcal{J} \to \mathcal{I}$ is Poincaré, see [CDH+20a, 6.3.18]

The cotensoring construction $(\mathcal{I}, (\mathcal{C}, \Omega)) \mapsto (\mathcal{C}, \Omega)^{\mathcal{I}}$ has the expected universal property, [CDH+20a, 6.3.10].

Proposition 2.32. *There is a canonical equivalence*

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{h}}_\infty}((\mathcal{C}, \Omega), (\mathcal{C}', \Omega')^{\mathcal{I}}) \simeq \mathrm{Core}(\mathrm{Fun}(\mathcal{I}, (\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}, \Omega), (\mathcal{C}', \Omega'))))).$$

Corollary 2.33. *There is a functor $\mathrm{Cat}^{\mathrm{op}}_\infty \times \mathrm{Cat}^{\mathrm{h}}_\infty \longrightarrow \mathrm{Cat}^{\mathrm{h}}_\infty$ sending $(\mathcal{I}, (\mathcal{C}, \Omega))$ to the cotensor $(\mathcal{C}, \Omega)^{\mathcal{I}}$.*

Proof. This follows from the Yoneda lemma and the fact that the formation of internal mapping objects is functorial by Corollary 2.23. □

2.3. **Grothendieck–Witt and L-spaces.** Let us now consider the following categories: $\mathcal{K}_n = \mathrm{TwArr}(\Delta^n)$, $\mathcal{I}_n \subseteq \mathcal{K}_n$ which is the subcategory on objects (i, j) with $j \leq i + 1$, and $\mathcal{T}_n$ the face poset of Δ^n. We note that also $\mathcal{I}_n$ and $\mathcal{T}_n$ are the face posets of simplicial complexes (of an interval with $n+1$ vertices in the former case and the n-simplex in the latter case).

With varying n, the categories $\mathcal{K}_n$ and $\mathcal{T}_n$ assemble into cosimplicial categories

$$\mathcal{K}_\bullet, \mathcal{T}_\bullet \colon \Delta \longrightarrow \mathrm{Cat}_\infty$$

and the induced maps $\mathcal{K}_n \to \mathcal{K}_m$ for a map $[n] \to [m]$ in Δ are all induced from simplicial maps between the simplicial complexes of which $\mathcal{K}_n$ and $\mathcal{K}_m$ are the face poset. The same is true for $\mathcal{T}_\bullet$.

Using the cotensoring construction and Proposition 2.30, we therefore obtain functors

$$\mathrm{Cat}^{\mathrm{p}}_\infty \longrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Cat}^{\mathrm{p}}_\infty)$$

given by

$$(\mathcal{C}, \Omega) \mapsto \left([n] \mapsto \begin{cases} (\mathcal{C}, \Omega)^{\mathcal{K}_n} \\ (\mathcal{C}, \Omega)^{\mathcal{T}_n} \end{cases}\right)$$

We note that there is a functor $\mathcal{I}_n \to \mathcal{K}_n$ which induces a functor $(\mathcal{C}, \Qoppa)^{\mathcal{K}_n} \to (\mathcal{C}, \Qoppa)^{\mathcal{I}_n}$ which, on underlying categories restricts a diagram

$$\begin{tikzcd}
 & & \bar{W} \arrow[dl] \arrow[dr] & & \\
 & W \arrow[dl] \arrow[dr] & & W' \arrow[dl] \arrow[dr] & \\
X & & X' & & X''
\end{tikzcd}$$

to the bottom two spans; this picture is the case $n = 2$, for higher n's, one similarly restricts a diagram to the n-composable spans on the bottom. This restriction functor admits a right adjoint, whose image consists exactly of the diagrams all whose squares are pullbacks.

It turns out that the hermitian structure $\Qoppa^{\mathcal{K}_n}$ on such diagrams coincides with the hermitian structure $\Qoppa^{\mathcal{I}_n}$ on the diagram obtained by restricting to the bottom spans.

Therefore, $(\mathcal{C}, \Qoppa)^{\mathcal{I}_n}$ can be identified with the full subcategory of $(\mathcal{C}, \Qoppa)^{\mathcal{K}_n}$ on diagrams all of whose squares are pullbacks, and is therefore also functorial in Δ (one simply has to note that restriction along the maps $\mathcal{K}_n \to \mathcal{K}_m$ induced from simplicial maps $[n] \to [m]$ preserve the property that a functor is right Kan extended from the subcategory $\mathcal{I}_n$ and $\mathcal{I}_m$).

Definition 2.34. We denote the resulting Poincaré category by $Q_n(\mathcal{C}, \Qoppa)$, and view it as a sub Poincaré category of $(\mathcal{C}, \Qoppa)^{\mathcal{K}_n}$. Likewise, we denote the Poincaré category $(\mathcal{C}, \Qoppa)^{\mathcal{T}_n}$ by $\rho_n(\mathcal{C}, \Qoppa)$, and refer to the resulting functors

$$Q_\bullet, \rho_\bullet \colon \mathrm{Cat}^{\mathrm{p}}_\infty \longrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Cat}^{\mathrm{p}}_\infty)$$

as the hermitian Q-construction and the hermitian ρ-construction.

We define the Grothendieck–Witt space[61] of $(\mathcal{C}, \Qoppa)$ as the loop space of the geometric realisation of the hermitian Q-construction of $(\mathcal{C}, \Sigma\Qoppa)$:

$$\mathrm{GW}(\mathcal{C}, \Qoppa) = \Omega|\mathrm{Poinc}(Q_n(\mathcal{C}, \Sigma\Qoppa))|$$

and the L-theory space of $(\mathcal{C}, \Qoppa)$ by the geometric realisation of the hermitian ρ-construction:

$$\mathrm{L}(\mathcal{C}, \Qoppa) = |\mathrm{Poinc}(\rho_n(\mathcal{C}, \Qoppa))|.$$

One might wonder at which basepoint we form the loops in the definition of GW. For this, we recall that Poinc is corepresented, so that the simplicial anima $\mathrm{Poinc}(Q_\bullet(\mathcal{C}, \Sigma\Qoppa))$ canonically refines to an object of

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{An}).$$

Since the forgetful functor $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{An}) \to \mathrm{An}$ preserves sifted colimits, we deduce that the geometric realisation $|\mathrm{Poinc}(Q_\bullet(\mathcal{C}, \Sigma\Qoppa))|$ is again canonically an $\mathbb{E}_\infty$-anima, and therefore comes with a preferred basepoint, which we use to form the loop space in the above definition.

Remark 2.35. Discarding Poincaré structures throughout, there are analogous constructions

$$Q_\bullet, \rho_\bullet \colon \mathrm{Cat}^{\mathrm{ex}}_\infty \longrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Cat}^{\mathrm{ex}}_\infty).$$

[61] Later, we will construct non-connective deloopings of this space, but we will refrain from giving the space and the spectrum different symbols. In [CDH+20b] the Grothendieck–Witt space is written as $\mathcal{GW}$ and the spectrum is written as GW.

which we refer to as the Q-construction and the ρ-construction. The above definition of the Grothendieck–Witt space is then analogous to the formula

$$K(\mathcal{C}) = \Omega|\mathrm{Core}(Q_n(\mathcal{C}))|,$$

and in addition, it is not hard to see that $|\mathrm{Core}(Q_n(\mathcal{C}))|$ is connected. On the L-theory side, we have that

$$|\mathrm{Core}(\rho_n(\mathcal{C}))| = *$$

is even contractible[62], so we will never mention the ρ-construction in stable categories, and hence also refer to the hermitian ρ-construction as the ρ-construction.

Exercise. Show that $\mathrm{GW}(\mathrm{Hyp}(\mathcal{C})) \simeq K(\mathcal{C})$.

Remark 2.36. It will soon become clear why we define $\mathrm{GW}(\mathcal{C}, \Omega\!\!\!\!\!\raisebox{0pt}{}\,)$ by a hermitian Q-construction associated to the shifted Poincaré structure $\Sigma Ϙ$. It is needed to ensure that $\pi_0(\mathrm{GW}(\mathcal{C}, Ϙ))$ is related to Poincaré objects in $(\mathcal{C}, Ϙ)$; the situation is similar as the shift one has in geometric cobordism categories: The category Cob_d has objects the closed $(d-1)$-manifolds and morphisms the d-dimensional cobordisms.

Remark 2.37. By construction, $\mathrm{GW}(\mathcal{C}, Ϙ)$ is a 1-fold loop space and therefore has a preferred π_{-1} for which we will see an explicit generators and relations description soon (it will be a cobordism group of Poincaré objects). We will in fact see later that *all* negative homotopy groups of the canonical non-connective delooping of $\mathrm{GW}(\mathcal{C}, Ϙ)$ are described as such cobordism groups. This features in the analysis of the fundamental fibre sequence relating GW to L-theory and K-theory.

2.4. **The cobordism category.** It will be useful to note that the Grothendieck–Witt space $\mathrm{GW}(\mathcal{C}, Ϙ)$ we have just constructed can be identified with the loop space of the anima associated to a cobordism ∞-category.

Informally, as noted before, let us perform the following construction.

Construction 2.38. Let $(\mathcal{C}, Ϙ)$ be a Poincaré category. Then there should be an ∞-category $\mathrm{Cob}(\mathcal{C}, Ϙ)$ associated to it which has as objects the Poincaré objects of $(\mathcal{C}, Ϙ)$ and where a morphism from (X, q) to (X', q') is a cobordism between them, i.e. a Poincaré object of $Q_1(\mathcal{C}, Ϙ)$. The composition of two cobordisms is the glued cobordism of Section 1.5 and identities are obtained by the constant cobordisms $(X = X = X)$.

A canonical way to make this informal description precise is to construct the cobordism ∞-category as a (complete) Segal anima. We briefly recall what those are and how they give rise to ∞-categories.

Recall that there is a canonical functor $\Delta \to \mathrm{Cat}_\infty$, which by the cocompleteness of Cat_∞ extends to a unique colimit preserving (in fact left adjoint functor) $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An}) \to \mathrm{Cat}_\infty$. Its right adjoint is given by

$$\mathcal{C} \mapsto ([n] \mapsto \mathrm{Core}(\mathrm{Fun}(\Delta^n, \mathcal{C}))).$$

Definition 2.39. A *Segal anima* is a functor $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathrm{An}$ such that the canonical maps

$$\mathcal{X}_n \longrightarrow \mathcal{X}_1 \times_{\mathcal{X}_0} \cdots \times_{\mathcal{X}_0} \mathcal{X}_1$$

are equivalences.

[62] We'll see one argument later, but it is also a good exercise to prove this directly.

We think of a Segal anima as an ∞-category with Core given by $\mathcal{X}_0$ and morphisms given $\mathcal{X}_1$. The simplicial structure maps $\mathcal{X}_1 \to \mathcal{X}_0$ give soure and target, the zig-zag

$$\mathcal{X}_1 \times_{\mathcal{X}_0} \mathcal{X}_1 \xleftarrow{\simeq} \mathcal{X}_2 \longrightarrow \mathcal{X}_1$$

defines a composition law, and the map $\mathcal{X}_0 \to \mathcal{X}_1$ defines units of objects.

One can then define internal equivalences $\mathrm{equiv}(\mathcal{X})$ of $\mathcal{X}$ as those points of $\mathcal{X}_1$ which admit left and right inverses with respect to the just explained composition product.

Definition 2.40. A Segal anima is called complete if the canonical map $\mathcal{X}_0 \to \mathrm{equiv}(\mathcal{X})$ is an equivalence, or equivalently if the diagram

$$\begin{tikzcd}
\mathcal{X}_0 \arrow[r] \arrow[d] & \mathcal{X}_3 \arrow[d] \\
\mathcal{X}_0 \times \mathcal{X}_0 \arrow[r] & \mathcal{X}_1 \times \mathcal{X}_1
\end{tikzcd}$$

is a pulback [CDH+20b, 9.1].

That is, for a complete Segal anima, the two notions of forming a Core of the category the Segal anima tries to describe coincide. This suggests that complete Segal anima are in fact closely related to ∞-categories, and indeed, we have the following theorem.

Theorem 2.41. *The above described functor* $\mathrm{Cat}_\infty \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})$ *is fully faithful, with essential image the complete Segal animae*[63]*. Its left adjoint is called the* associated category functor *and will be written* $\mathrm{asscat}(-)$[64].

The relation between ∞-categories and complete Segal anima is due to Rezk, and also most of the following remarks are due to Rezk.

Remark 2.42. The inclusion $\mathrm{An} \subseteq \mathrm{Cat}_\infty$ has both adjoints: The right adjoint is Core and the left adjoint is given by inverting all morphisms denoted by $|-|$, as it turns out to be the composite

$$\mathrm{Cat}_\infty \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An}) \xrightarrow{\mathrm{colim}} \mathrm{An}.$$

In addition, the completion functor induces an equivalence $|\mathrm{asscat}(\mathcal{X})| \simeq |\mathcal{X}|$ for any simplicial anima $\mathcal{X}$[65].

Remark 2.43. For a simplicial anima $\mathcal{X}$ there is a canonical map $\mathcal{X}_0 \to \mathrm{Core}(\mathrm{asscat}(\mathcal{X}))$ which always surjective on π_0 [Lur09b, 1.2.17] and an equivalence if $\mathcal{X}$ is complete.

Remark 2.44. The completion functor from Segal animae to complete Segal animae (in other words, the functor asscat restricted to Segal animae) preserves finite products.

[63] Exercise: Show that the functor indeed takes values in complete Segal animae.

[64] We will not distinguish between complete Segal animae and ∞-categories from now on. Therefore asscat sometimes refers to an ∞-category, and sometimes to a complete Segal animae. I hope it will always be clear from the context what is meant, and hope that this abuse of notation will not lead to confusion.

[65] Exercise: Prove these assertions.

Remark 2.45. From a simplicial anima $\mathcal{X}$ and objects $X_0, X_1 \in \mathcal{X}_0$ we can extract an anima of maps $\mathrm{Map}_{\mathcal{X}}(X_0, X_1)$ from X_0 to X_1 defined by the pullback

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{X}}(X_0,X_1) \arrow[r] \arrow[d] & \mathcal{X}_1 \arrow[d] \\ * \arrow[r, "{(X_0,X_1)}"] & \mathcal{X}_0 \times \mathcal{X}_0 \end{tikzcd}$$

This recovers the usual definition for ∞-categories, when viewed as complete Segal animae. The map $\mathcal{X} \to \mathrm{asscat}(\mathcal{X})$, induces an equivalence on animae of maps [Lur09b, 1.2.27] provided $\mathcal{X}$ is a Segal anima.

With this in mind we have the following proposition.

Proposition 2.46. *The simplicial Poincaré category $Q_\bullet(\mathcal{C}, \Omega)$ is a Segal object, and is complete in the sense that the diagram*

$$\begin{tikzcd} (\mathcal{C},\Omega) \arrow[r] \arrow[d] & Q_3(\mathcal{C},\Omega) \arrow[d] \\ (\mathcal{C},\Omega)\times(\mathcal{C},\Omega) \arrow[r] & Q_1(\mathcal{C},\Omega)\times Q_1(\mathcal{C},\Omega) \end{tikzcd}$$

Corollary 2.47. *Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{An}$ be an additive functor. Then $\mathcal{F}(Q_\bullet(\mathcal{C}, \Omega))$ is a Segal anima which is complete if $\mathcal{F}$ preserves pullbacks*[66]*, for instance for $\mathcal{F} = \mathrm{Poinc}$.*

Definition 2.48. We denote by $\mathrm{Cob}(\mathcal{C}, \Omega)$ the (∞-category associated to the) complete Segal anima $\mathrm{Poinc}(Q_\bullet(\mathcal{C}, \Sigma\Omega))$.

Remark 2.49. We therefore find that $\mathrm{GW}(\mathcal{C}, \Omega) = \Omega|\mathrm{Cob}(\mathcal{C}, \Omega)|$.

Exercise. Show that $\pi_0(\mathrm{Cob}(\mathcal{C}, \Omega))$ is canonically isomorphic to the cobordism group $L_0(\mathcal{C}, \Sigma\Omega)$ of Definition 1.65.

Remark 2.50. For an additive functor $\mathcal{F}$, the Segal anima $\mathcal{F}(Q_\bullet(\mathcal{C}, \Omega))$ is not complete in general. In fact, if $\mathcal{F}$ is group-like then this Segal anima is complete if and only if $\mathcal{F}(\mathrm{Hyp}(\mathcal{C})) = 0$[67].

Nevertheless, we can make the following definition.

Definition 2.51. For an additive functor $\mathcal{F}$, we let $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Omega)$ be (the ∞-category associated to) the completion of the Segal anima $\mathcal{F}(Q_\bullet(\mathcal{C}, \Sigma\Omega))$.

Remark 2.52. We may apply this to the functor Core (viewed either on $\mathrm{Cat}^{\mathrm{p}}_\infty$ or $\mathrm{Cat}^{\mathrm{ex}}_\infty$). Then we obtain that $\mathrm{Cob}^{\mathrm{Core}}(\mathcal{C}) = \mathrm{Span}(\mathcal{C})$ is just the usual span category.

Our aim is to show that the association $(\mathcal{C}, \Omega) \to \mathrm{GW}(\mathcal{C}, \Omega)$ receives a map from Poinc, that this map satisfies a universal property, and use this to give a presentation of $\pi_0(\mathrm{GW}(\mathcal{C}, \Omega))$.

We observe that there is a canonical map

$$\mathrm{Map}_{\mathrm{Cob}(\mathcal{C},\Omega)}(0,0) \longrightarrow \mathrm{Map}_{|\mathrm{Cob}(\mathcal{C},\Omega)|}(0,0) \simeq \Omega|\mathrm{Cob}(\mathcal{C},\Omega)| = \mathrm{GW}(\mathcal{C},\Omega).$$

[66] Note that an additive functor preserves certain types of pullbacks, but not all in general

[67] We might come to this later, else see [CDH+20b, 10.2.18]

Considering the commutative diagram[68]

$$\begin{tikzcd}
(\mathcal{C}, \Omega) \arrow[r] \arrow[d] & Q_1(\mathcal{C}, \Sigma\Omega) \arrow[d] \\
0 \arrow[r, "{(0,0)}"] & (\mathcal{C}, \Sigma\Omega) \times (\mathcal{C}, \Sigma\Omega)
\end{tikzcd}$$

where the upper horizontal functor sends X to the span $(0 \leftarrow X \rightarrow 0)$.

Applying Poinc to this diagram gives a commutative diagram, which by the definition of mapping anima associated to a Segal space gives a canonical map

$$\mathrm{Poinc}(\mathcal{C}, \Omega) \longrightarrow \Omega|\mathrm{Cob}(\mathcal{C}, \Sigma\Omega)| = \mathrm{GW}(\mathcal{C}, \Omega).$$

The same argument (use that any simplicial anima maps to its associated complete Segal anima) gives more generally for any functor $\mathcal{F}\colon \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathrm{An}$, a canonical map

$$\mathcal{F}(\mathcal{C}, \Omega) \longrightarrow \Omega|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Omega)|.$$

2.5. **The additivity theorem.** The theorem we now want to prove is the following:

Theorem 2.53. *Let $\mathcal{F}$ be an additive functor. Then the functor $|\mathrm{Cob}^{\mathcal{F}}(-)|$ is again additive. Therefore, the functor $\Omega|\mathrm{Cob}^{\mathcal{F}}(-)|$ is additive and group-like. If $\mathcal{F}$ is group-like, then the map*

$$\mathcal{F} \longrightarrow \Omega|\mathrm{Cob}^{\mathcal{F}}(-)|$$

is an equivalence.

Remark 2.54. Informally, this theorem says that the association $\mathcal{F} \mapsto \Omega|\mathrm{Cob}^{\mathcal{F}}(-)|$ is the left adjoint to the inclusion of group-like additive into additive functors: Given a map $\mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ group-like additive, we consider the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d, "\simeq"] \\
\Omega|\mathrm{Cob}^{\mathcal{F}}(-)| \arrow[r] \arrow[ur, dashed] & \Omega|\mathrm{Cob}^{\mathcal{G}}(-)|
\end{tikzcd}$$

and inverting the right vertical map, we get an essentially unique extension $\Omega|\mathrm{Cob}^{\mathcal{F}}(-)| \to \mathcal{G}$.

More formally[69], we would like to show that the above association is a localisation L of $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{p}}, \mathrm{An})$. The above map induces a natural transformation $\eta\colon \mathrm{id} \to L$, and in order to show that L is in fact a localisation, we need to prove that the *two* canonical maps $\eta_{LX}, L\eta_X\colon LX \to L^2X$ are equivalences. The theorem above only gives that one of the two is. It turns out, however, that this formally implies, in our situation, that also the second map is an equivalence. This is because it turns out that the association $\mathcal{F} \mapsto |\mathrm{Cob}^{\mathcal{F}}(-)|$ can be identified with the suspension functor of the category $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{p}}, \mathrm{An})$ – we will indicate why this is so later. The above transformation is the the unit of the suspension-loops

[68] In fact, this is a pullback diagram

[69] It still needs to be shown that this extension is itself functorial.

adjunction[70]. In this case, the multiplication map of the monad $\Omega\Sigma$ gives a common section of the two maps $\Omega\Sigma \to \Omega\Sigma\Omega\Sigma$, so one is an equivalence if and only if the other is[71].

Remark 2.55. Specialising the above to $\mathcal{F} = \mathrm{Poinc}$, we find that GW is the initial group-like additive functor equipped with a map from Poinc.

The proof consists of two in fact quite separate arguments. On the one hand, we must show that the association $\mathcal{F} \mapsto |\mathrm{Cob}^{\mathcal{F}}(-)|$ preserves additive functors, and on the other hand, we must show that if $\mathcal{F}$ was already group-like, this operation does not change the input (up to canonical equivalence).

We will first show this second part:

Proposition 2.56. *Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An})$ be a group-like additive functor. Then the map*

$$\mathcal{F}(\mathcal{C}, Ϙ) \longrightarrow \Omega|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)|$$

is an equivalence.

Proof. We consider the sequence

$$\mathrm{Met}(\mathcal{C}, Ϙ) \longrightarrow Q_1(\mathcal{C}, Ϙ) \longrightarrow (\mathcal{C}, Ϙ)$$

where the target is given by the evaluation of either of the two "boundaries" and where the first map is the inclusion of the kernel of this map. As we have observed in Example 2.9, this is a split PV sequence, which in addition admits a splitting through Poincaré functors given by sending X to the constant span $(X = X = X)$. We deduce that applying $\mathcal{F}$ to the above sequence, we obtain a preferred equivalence[72]

$$\mathcal{F}(Q_1(\mathcal{C}, Ϙ) \simeq \mathcal{F}(\mathrm{Met}(\mathcal{C}, Ϙ)) \oplus \mathcal{F}(\mathcal{C}, Ϙ).$$

Similarly, we recall from Proposition 2.46, that $Q_\bullet(\mathcal{C}, Ϙ)$ is a Segal object. For instance, we have a pullback diagram

$$\begin{tikzcd} Q_2(\mathcal{C}, Ϙ) \arrow[r] \arrow[d] & Q_1(\mathcal{C}, Ϙ) \arrow[d] \\ Q_1(\mathcal{C}, Ϙ) \arrow[r] & (\mathcal{C}, Ϙ) \end{tikzcd}$$

where the two maps to $(\mathcal{C}, Ϙ)$ are evaluation at the left and right boundary, respectively. In particular, they are split PV projections. We deduce that the above square is a split PV square[73] and therefore upon applying $\mathcal{F}$, we obtain a pullback. Using the above formula for $\mathcal{F}(Q_1(\mathcal{C}, Ϙ))$ we then see that there is a canonical equivalence

$$\mathcal{F}(Q_2(\mathcal{C}, Ϙ)) \simeq \mathcal{F}(\mathrm{Met}(\mathcal{C}, Ϙ)) \oplus \mathcal{F}(\mathrm{Met}(\mathcal{C}, Ϙ)) \oplus \mathcal{F}(\mathcal{C}, Ϙ).$$

Inductively, we deduce from the Segalness of $Q_\bullet(\mathcal{C}, Ϙ)$ that there is an equivalence

$$\mathcal{F}(Q_n(\mathcal{C}, Ϙ)) \simeq \mathcal{F}(\mathrm{Met}(\mathcal{C}, Ϙ))^{\oplus n} \oplus \mathcal{F}(\mathcal{C}, Ϙ)$$

[70]Notice that limits in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathrm{An})$ are calculated pointwise, but colimits are *not* computed pointwise, is pointwise colimits don't preserve additivity of functors

[71]One can also show that the second map is merely a "coordinate change" of the first map, so one obtains the adjunction result also without having to show that the formation of $|\mathrm{Cob}^{\mathcal{F}}(-)|$ is the suspension operation on additive functors.

[72]The map from the right to the left is induced by the inclusion and the split

[73]Exercise: Show that split PV sequences are closed under pullbacks.

which identifies the simplicial anima $\mathcal{F}(Q_\bullet(\mathcal{C}, Ϙ))$ with the action groupoid associated to the map $\mathcal{F}(\mathrm{Met}(\mathcal{C}, Ϙ)) \to \mathcal{F}(\mathcal{C}, Ϙ)$ of $\mathbb{E}_\infty$-groups induced by the map $\mathrm{Met}(\mathcal{C}, Ϙ) \to (\mathcal{C}, Ϙ)$ discussed in Lemma 1.60[74].

It follows that the geometric realisation $|\mathcal{F}(Q_\bullet(\mathcal{C}, Ϙ))|$ identifies with the *cofibre* in $\mathbb{E}_\infty$-groups of the map

$$\mathcal{F}(\mathrm{Met}(\mathcal{C}, Ϙ)) \longrightarrow \mathcal{F}(\mathcal{C}, Ϙ).$$

Now, in $\mathbb{E}_\infty$-groups, cofibres are calculated by taking the suspension of the fibre. However, the fibre can be calculated by means of the split PV sequence[75]

$$(\mathcal{C}, \Omega Ϙ) \longrightarrow \mathrm{Met}(\mathcal{C}, Ϙ) \longrightarrow (\mathcal{C}, Ϙ).$$

In summary, we obtain an equivalence

$$|\mathcal{F}(Q_\bullet(\mathcal{C}, Ϙ))| \simeq \Sigma\mathcal{F}(\mathcal{C}, \Omega Ϙ).$$

This identification is made such that the following holds: Applying this for $\Sigma Ϙ$ in place of $Ϙ$, and looping this equivalence once gives an equivalence

$$\mathcal{F}(\mathcal{C}, Ϙ) \simeq \Omega|\mathcal{F}(Q_\bullet(\mathcal{C}, \Sigma Ϙ))| = \Omega|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)$$

which coincides with the map we have constructed earlier. The proposition follows. □

Remark 2.57. Since $\mathcal{F}(\mathrm{Met}(\mathcal{C}, Ϙ)) \simeq \mathcal{F}(\mathrm{Hyp}(\mathcal{C}))$ for a group-like additive functor, we also obtain that $\mathcal{F}(Q_\bullet(\mathcal{C}, Ϙ))$ is the action groupoid for the $\mathbb{E}_\infty$-map $\mathrm{Hyp}(\mathcal{C}) \to (\mathcal{C}, Ϙ)$. One might wonder how this fits together, as the diagram

$$\begin{tikzcd} \mathrm{Met}(\mathcal{C}, Ϙ) \arrow[r, "\mathrm{lag}"] \arrow[dr] & \mathrm{Hyp}(\mathcal{C}) \arrow[d] \\ & (\mathcal{C}, Ϙ) \end{tikzcd}$$

does *not* commute. However, it commutes after applying any group-like additive functor $\mathcal{F}$: After applying such, the functor lag is inverse to the functor $\mathrm{Hyp}(\mathcal{C}) \to \mathrm{Met}(\mathcal{C}, Ϙ)$ induced, by adjunction, by the functor sending X to $(X = X)$: This is simply because $\mathcal{F}(\mathrm{lag})$ is an equivalence and the composite $\mathrm{Hyp}(\mathcal{C}) \to \mathrm{Met}(\mathcal{C}, Ϙ) \to \mathrm{Hyp}(\mathcal{C})$ is the identity. Hence replacing lag with this Poincaré functor, the above diagram commute already on the level of Poincaré categories.

This has the advantage that $\mathcal{F}(\mathrm{Hyp}(\mathcal{C}))$ is typically easier to evaluate (from its definition) that $\mathcal{F}(\mathrm{Met}(\mathcal{C}, Ϙ))$. For instance for $\mathcal{F}$ being K-theory of the underlying category, we obtain $K(\mathrm{Hyp}(\mathcal{C})) = K(\mathcal{C}) \oplus K(\mathcal{C})$ without having to use additivity for K-theory.

Exercise. Show that $\mathcal{F}(Q_\bullet(\mathcal{C}, Ϙ))$ is complete if and only if $\mathcal{F}(\mathrm{Met}(\mathcal{C}, Ϙ))$, or equivalently $\mathcal{F}(\mathrm{Hyp}(\mathcal{C}))$ is trivial. (Or more generally, show that the action groupoid object of a map of $\mathbb{E}_\infty$-groups $Y \to X$ is complete if and only if $Y = 0$.)

Next we will sketch the additivity theorem:

Proposition 2.58. *Let* $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{An}$ *be an additive functor. Then* $|\mathrm{Cob}^{\mathcal{F}}(-)|$ *is again additive.*

[74]Exercise: Show that this is in fact the case.
[75]Exercise: Show this sequence is indeed split PV

Proof. To prove the proposition, we take two steps. We will show that for a split PV projection $p\colon (\mathcal{C}, Ϙ) \to (\mathcal{D}, \bar{Ϙ})$, the induced map

$$\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ) \longrightarrow \mathrm{Cob}^{\mathcal{F}}(\mathcal{D}, \bar{Ϙ})$$

is a bicartesian fibration of ∞-categories. In fact it suffices to show that it is a cocartesian fibration, as this implies that the map is also a cartesian fibration, by making use of the dualities on $\mathcal{C}$ and $\mathcal{D}$, and the fact that p preserves the dualities, but the proof we indicate here will give the cartesian and cocartesian case at the same time.

Taking this statement for granted for the moment, let us consider a split PV square as follows.

$$\begin{tikzcd} (\mathcal{C}', Ϙ') \arrow[r] \arrow[d] & (\mathcal{C}, Ϙ) \arrow[d] \\ (\mathcal{D}', \bar{Ϙ}') \arrow[r] & (\mathcal{D}, \bar{Ϙ}) \end{tikzcd}$$

We then show that the induced diagram

$$\begin{tikzcd} \mathrm{Cob}^{\mathcal{F}}(\mathcal{C}', Ϙ') \arrow[r] \arrow[d] & \mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ) \arrow[d] \\ \mathrm{Cob}^{\mathcal{F}}(\mathcal{D}', \bar{Ϙ}') \arrow[r] & \mathrm{Cob}^{\mathcal{F}}(\mathcal{D}, \bar{Ϙ}) \end{tikzcd}$$

is a pullback square, and note that its right vertical leg is a bicartesian fibration of ∞-categories. In Proposition 2.60 we will show that such squares stay pullback squares after applying geometric realisations, showing that $|\mathrm{Cob}^{\mathcal{F}}(-)|$ is an additive functor as needed.

To see that the square of cobordism categories is in fact a pullback, we note that applying $\mathcal{F}$ to the Q-construction applied to the split PV square in question gives a levelwise pullback of simplicial anima. This is because Q_n preserves split PV squares and $\mathcal{F}$ is additive. Therefore we know that $\mathcal{F}(Q_\bullet(-))$ of the diagram in question is a pullback in Segal anima (the diagram is one of Segal anima and they are closed under pullbacks in all simplicial anima). Now in general, the completion functor does not preserve pullbacks. However, we obtain a canonical functor

$$\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}', Ϙ') \longrightarrow \mathrm{Cob}^{\mathcal{F}}(\mathcal{D}', \bar{Ϙ}') \times_{\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)} \mathrm{Cob}^{\mathcal{F}}(\mathcal{D}, \bar{Ϙ})$$

which is fully faithful (since the completion functor does not change mapping anima on Segal objects and all simplicial objects in question are Segal). The problem is to see whether it is also an essentially surjective functor, but this can be deduced from the fact that the map $\mathcal{F}(Q_\bullet(\mathcal{C}, Ϙ)) \to \mathcal{F}(Q_\bullet(\mathcal{D}, \bar{Ϙ}))$ is an isofibration (i.e. that one has a certain lifting property for equivalences). See [CDH+20b, proof of the additivity theorem].

Now, we recall again that $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)$ is the completion of the Segal anima $\mathcal{F}(Q_\bullet(\mathcal{C}, Ϙ))$ and that the completion functor preserves mapping anima. Therefore we can directly extract the mapping anima of $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)$; they are given as pullbacks of $\mathcal{F}(Q_1(\mathcal{C}, Ϙ))$ over certain points in $\mathcal{F}(\mathcal{C}, Ϙ) \times \mathcal{F}(\mathcal{C}, Ϙ)$. One can consider the subcategory $\mathcal{E}$ of $Q_1(\mathcal{C}, Ϙ)$ on spans

$$(X' \xleftarrow{f} W \xrightarrow{g} X)$$

where f is p-cartesian and g is p-cocartesian. One checks that this subcategory is closed under the duality in $Q_1(\mathcal{C}, \Omega)$. The claim is that $\mathcal{F}(\mathcal{E})$ consists of $\mathrm{Cob}^{\mathcal{F}}(p)$-cocartesian edges.[76] ☐

We now work towards an ∞-categorical version of Quillen's Theorem B. First, we record the following lemma.

Lemma 2.59. *Let $\mathcal{C}$ be an ∞-category and $|\mathcal{C}| \to \mathrm{An}$ be a functor. Then in the pullback diagram*

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & \bar{\mathcal{E}} \arrow[d] \\ \mathcal{C} \arrow[r] & {|\mathcal{C}|} \end{tikzcd}$$

where $\bar{\mathcal{E}} \to |\mathcal{C}|$ is the right (and hence Kan) fibration associated to the functor $|\mathcal{C}| \to \mathrm{An}$, the map $\mathcal{E} \to \bar{\mathcal{E}}$ exhibits $\bar{\mathcal{E}}$ as $|\mathcal{E}|$.

Proof. This follows for instance because a standard construction of the map $\mathcal{C} \to |\mathcal{C}|$ shows it to be anodyne and an anodyne map pulled back along a Kan fibration is again anodyne (see e.g. [Lan21, 4.4.16] for the version with right anodyne maps and left fibrations whose dual version for left anodyne and right fibrations formally implies our needed result). Therefore $\mathcal{E} \to \bar{\mathcal{E}}$ is anodyne and $\bar{\mathcal{E}}$ is an ∞-groupoid (as it comes with a conservative functor to the ∞-groupoid $|\mathcal{C}|$). ☐

Proposition 2.60. *Let*

$$\begin{tikzcd} \mathcal{E}' \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{C}' \arrow[r, "g"] & \mathcal{C} \end{tikzcd}$$

be a pullback diagram of ∞-categories with $p\colon \mathcal{E} \to \mathcal{C}$ a cocartesian fibration. Assume that for each morphism $f\colon x \to y$ in $\mathcal{C}$, the induced functor $\mathcal{E}_x \to \mathcal{E}_y$ induces an equivalence after geometric realisation. Then the diagram obtained from the above pullback by applying geometric realisations is again a pullback.

Proof. Let $F\colon \mathcal{C} \to \mathrm{Cat}_\infty$ be the functor classifying the cocartesian fibration p. Consider the functor F' obtained from F by post composition with the geometric realisation functor $\mathrm{Cat}_\infty \to \mathrm{An}$. To it is associated a left fibration over $\mathcal{C}$, and it is the functor $\mathcal{E}[\mathrm{fibre}^{-1}] \to \mathcal{C}$, where fibre is the collection of morphisms of $\mathcal{E}$ which lie over an identity of an object in $\mathcal{C}$. Therefore, we obtain pullback diagram

$$\begin{tikzcd} \mathcal{E}'[\mathrm{fibre}^{-1}] \arrow[r] \arrow[d] & \mathcal{E}[\mathrm{fibre}^{-1}] \arrow[d] \\ \mathcal{C}' \arrow[r] & \mathcal{C} \end{tikzcd}$$

The assumption that the map $\mathcal{E}_x \to \mathcal{E}_y$ induce equivalences after geometric realisation says that the functor $\mathcal{C} \to \mathrm{Cat}_\infty \to \mathrm{An}$ factors through the canonical map $\mathcal{C} \to |\mathcal{C}|$. It follows that

[76] For a formal argument, see [CDH+20b, 9.5.3], but I recommend to work out the universal property on mapping anima in an informal way.

the restriction to $\mathcal{C}'$ along g also factors through $\mathcal{C}' \to |\mathcal{C}'|$ so that we have a diagram pullback diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ |\mathcal{C}'| \arrow[r, "|g|"] & |\mathcal{C}| \end{tikzcd}$$

where $\mathcal{X}$ is the left fibration associated to the functor $|\mathcal{C}| \to \mathrm{An}$ and $\mathcal{X}'$. But now Lemma 2.59 shows that $\mathcal{X}$ is canonically equivalent to $|\mathcal{E}[\mathrm{fibre}^{-1}]|$, and likewise that $\mathcal{X}'$ is canonically equivalent to $|\mathcal{E}'[\mathrm{fibre}^{-1}]|$. The claim the follows from the observation that the map $\mathcal{E} \to \mathcal{E}[\mathrm{fibre}^{-1}]$ induces an equivalence on geometric realisation. We recommend to do this as an exercise: for any set of maps W of an ∞-category $\mathcal{D}$, the canonical map $\mathcal{D} \to \mathcal{D}[W^{-1}]$ induces an equivalence $\mathcal{D}[\mathrm{all}^{-1}] \to (\mathcal{D}[W^{-1}])[\mathrm{all}^{-1}]$. Note, however, that in general it is not true that $\mathcal{D}[W^{-1}][V^{-1}]$ is a Dwyer–Kan localisation of $\mathcal{D}$. □

Finally, we note that a bicartesian fibration satisfies the assumption on the fibre functors appearing in Proposition 2.60, since the fibre functors then have adjoints. After passing to geometric realisations, these become equivalences as needed.

For completeness we indicate a sketch of the proof the following proposition:

Proposition 2.61. $|\mathrm{Cob}^{\mathcal{F}}(-)|$ *is the suspension of* $\mathcal{F}$ *is the category of additive anima-valued functors.*

Proof. First, we introduce some notation. Consider the functor $\mathrm{dec}\colon \Delta \to \Delta$ given by sending $[n]$ to $[1+n]$. We denote the restriction of a simplicial object $\mathcal{X}$ along dec by $\mathrm{dec}(\mathcal{X})$. Evaluation at zero gives a transformation $\mathrm{dec}(\mathcal{X}) \to \mathrm{const}(\mathcal{X}_0)$, making dec an augmented or split simplicial object. It follows that the augmentation induces an equivalence on geometric realisations

$$|\mathrm{dec}(\mathcal{X})| \xrightarrow{\simeq} |\mathrm{const}(\mathcal{X}_0)| \simeq \mathcal{X}_0.$$

Omitting the symbol const, when viewing objects as simplicial objects, we define $\mathrm{Null}_\bullet(\mathcal{C}, \Qoppa)$ as the pullback

$$\begin{tikzcd} \mathrm{Null}_\bullet(\mathcal{C}, \Qoppa) \arrow[r] \arrow[d] & \mathrm{dec}(Q_\bullet(\mathcal{C}, \Qoppa)) \arrow[d] \\ 0 \arrow[r] & (\mathcal{C}, \Qoppa) \end{tikzcd}$$

exhibiting $\mathrm{Null}_\bullet(\mathcal{C}, \Qoppa)$ as augmented over 0. Applying $\mathcal{F}$, we deduce that $|\mathcal{F}(\mathrm{Null}_\bullet(\mathcal{C}, \Qoppa))| \simeq F(0) = 0$. Now, the map dec receives a natural transformation from id (given by the map $d_0\colon [n] \to [1+n]$), which induces a canonical simplicial map $\mathrm{Null}_\bullet(\mathcal{C}, \Qoppa) \to Q_\bullet(\mathcal{C}, \Qoppa)$.

Inspecting the definitions, we find the following, see [CDH+20b, 10.3.3].

Lemma 2.62. *For every* $[n]$, $\mathrm{Null}_n(\mathcal{C}, \Qoppa) \to Q_n(\mathcal{C}, \Qoppa)$ *is a split PV projection, with kernel* $(\mathcal{C}, \Omega\Qoppa)$.

The case $n = 0$ of the above lemma is just the Bott-Genauer sequence

$$(\mathcal{C}, \Omega\Qoppa) \longrightarrow \mathrm{Met}(\mathcal{C}, \Qoppa) \longrightarrow (\mathcal{C}, \Qoppa)$$

we have seen earlier.