THE INITIAL QUADRATIC FORM ON AN ABELIAN GROUP

FABIAN HEBESTREIT, MARKUS LAND, AND THOMAS NIKOLAUS

The published version of our paper *On the homotopy of L-spectra of the integers* contains the following assertion as [HLN21, Remark A.3 ii)]:

For any abelian group A the map

$$\Delta_A \colon A \longrightarrow (A \otimes A)^{\mathrm{C}_2} \qquad a \longmapsto a \otimes a$$

is quadratic; in fact, one can show with some work that this is the initial such quadratic morphism, i.e. that $(A \otimes A)^{\mathrm{C}_2}$ corepresents the functor, which sends an abelian group S to quadratic forms on A with values in S.

While the map in question certainly is quadratic, it is *not* the initial such in general. Here, we call a map $q \colon A \to S$ quadratic if

(1) $q(ra) = r^2 q(a)$ for all $r \in \mathbb{Z}$ and $a \in A$, and
(2) $q(ra) = r^2 q(a)$ for all $r \in \mathbb{Z}$ and $a \in A$, and its polarisation

$$b_q \colon A \times A \longrightarrow S \quad b_q(a,b) = q(a+b) - q(a) - q(b)$$

is bilinear.

Embarrassingly, the assertion above already fails for $A = \mathbb{Z}/2$: The function

$$\mathbb{Z}/2 \longrightarrow \mathbb{Z}/4, \quad 0 \longmapsto 0, 1 \longmapsto 1$$

is easily checked quadratic, but $\Delta_{\mathbb{Z}/2} \colon \mathbb{Z}/2 \to (\mathbb{Z}/2 \otimes \mathbb{Z}/2)^{\mathrm{C}_2}$ is in fact a linear isomorphism, and the present form can clearly not be obtained by postcomposing $\Delta_{\mathbb{Z}/2}$ with a linear map to $\mathbb{Z}/4$.

We thank Emanuele Dotto and Yonatan Harpaz for bringing the mistake to our attention. By way of apology we derive the following correction in this note; it has no effect on any other statement in [HLN21].

Correction. *The map Δ_A is the initial quadratic form on A if and only if every 2-power torsion element in A is divisible by 2. Generally, it is initial among all quadratic maps q on A for which $2^k q(a) = 0$, whenever $2^k a = 0$.*

The initial quadratic form on a general abelian group A is indeed more complicated. Its target was originally introduced by Whitehead in [Whi50, Chapter 2] under the name $\Gamma(A)$, and we shall describe its precise relation to $(A \otimes A)^{\mathrm{C}_2}$ below.

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The initial quadratic form in general. The fact that there is an initial quadratic form at all is of course entirely formal: As its target one can simply take the free abelian group on symbols $q(a)$ and $b(a,a')$ for $a,a' \in A$ and divide by the relations given above (which in particular makes the second kind of generators superfluous). Using this description of $\Gamma(A)$ most of what we say below is either contained in, or can quickly be deduced from, the treatment in [EML54, Section 13], see also [Bau94, Section 2]. We will give a more concrete construction of $\Gamma(A)$ based on [CDH$^+$20, Remark 4.2.24], that is slightly easier to analyse directly: Denote by $W(A)$ the abelian group with underlying set $A \times (A \otimes A)_{C_2}$ and addition

$$(a,u) + (b,v) = (a+b, u+v-[a\otimes b]),$$

inverses are given by $-(a,v) = (-a, -v-[a\otimes a])$, and define $\Gamma(A)$ the quotient group given by identifying $(a,0)$ and $(-a,0)$ for all $a \in A$. One quickly computes

$$(a,0) - (-a,0) = (a,0) + (a, -[a\otimes a]) = (2a, -2[a\otimes a])$$

and it is easy to check that there results a short exact sequence

$$A/A[2] \xrightarrow{a\mapsto(2a,-2[a\otimes a])} W(A) \longrightarrow \Gamma(A).$$

The function

$$q_A \colon A \longrightarrow \Gamma(A), \quad a \longmapsto [a,0]$$

is then a quadratic form as defined above, with polarisation $b_A(a,b) = [0,[a\otimes b]]$: For the verification let $i_A \colon A \to W(A)$ send a to $(a,0)$. The claim is implied by

$$\begin{aligned} i_A(a+b) - i_A(a) - i_A(b) &= (a+b,0) + (-a,-[a\otimes a]) + (-b,-[b\otimes b]) \\ &= (b,[b\otimes a]) + (-b,-[b\otimes b]) \\ &= (0,[a\otimes b]) \end{aligned}$$

and

$$\begin{aligned} r^2 i_A(a) - i_A(ra) &= r^2(a,0) - (ra,0) \\ &= \left(r^2 a, \frac{r^2(1-r^2)}{2}[a\otimes a]\right) + \left(-ra, -r^2[a\otimes a]\right) \\ &= \left(r(r-1)a, \frac{r^2(1-r^2) - 2r^2 + 2r^3}{2}[a\otimes a]\right) \\ &= \left(2\cdot\frac{r(r-1)}{2}a, 0\right) + \left(0, \frac{-r^2(r-1)^2}{2}[a\otimes a]\right) \\ &\sim \left(0, 2\cdot\left[\frac{r(r-1)}{2}a \otimes \frac{r(r-1)}{2}a\right]\right) + \left(0, \frac{-r^2(r-1)^2}{2}[a\otimes a]\right) \\ &= 0. \end{aligned}$$

Given now a map $q\colon A \to S$ with bilinear polarisation, the first of these calculations shows that we obtain an additive map

$$\phi_q \colon W(A) \longrightarrow S, \quad (a,u) \longmapsto q(a) + b_q(u),$$

where we regard the polarisation of q as a map $(A\otimes A)_{C_2} \to S$. It factors over $\Gamma(A)$ if and only if q also satisfies the condition (1) above: ϕ_q factoring over $\Gamma(A)$ translates to $q(a) = q(-a)$ for all $a \in A$ which is clearly a consequence of the quadraticity, and the converse follows from the quadraticity of q_A.

Furthermore, we have $\phi_q \circ q_A = q$ per definition for any such q and for any homomorphism $f\colon \Gamma(A) \to S$ we compute

$$\phi_{f \circ q_A}[a,u] = f(q_A(a)) + f(b_A(u)) = f[a,0] + f[0,u] = f([a,0]+[0,u]) = f[a,u].$$

In total this shows that

$$\mathrm{Hom}(\Gamma(A), S) \longrightarrow \mathrm{Quad}(A,S), \quad f \longmapsto f \circ q_A$$

is bijective with inverse $q \mapsto \phi_q$. Since it is also clearly natural in S, we conclude that q_A is the desired initial quadratic map.

In particular, our $\Gamma(A)$ agrees with that defined by Whitehead.

Remark. (1) *The second calculation above shows that given the bilinearity of the polarisation of some map q, its quadraticity is in fact equivalent to the assertion $q(-a) = q(a)$ and also to $q(2a) = 2b_q(a,a)$, which encodes $q(2a) = 4q(a)$.*

(2) *A simple direct computation shows that $\mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2^{k-1} \cong W(\mathbb{Z}/2^k)$ for $k \geq 1$, the two subgroups generated by $(1,0)$ and $(2,2)$, respectively. Thus $\Gamma(\mathbb{Z}/2^k) = \mathbb{Z}/2^{k+1}$ and in particular the quadratic form $\mathbb{Z}/2 \to \mathbb{Z}/4$, that we gave as a counterexample to our original assertion above, is in fact the initial one on $\mathbb{Z}/2$.*

The relation to $(A \otimes A)^{\mathrm{C}_2}$. The quadratic map $\Delta_A\colon A \to (A \otimes A)^{\mathrm{C}_2}, a \mapsto a \otimes a$, is classified by

$$p_A\colon \Gamma(A) \longrightarrow (A \otimes A)^{\mathrm{C}_2}, \quad [a,u] \longmapsto (a \otimes a) + \mathrm{nm}_A(u),$$

which fits into a commutative diagram

$$\begin{array}{ccc} (A \otimes A)_{\mathrm{C}_2} & \xrightarrow{\quad \mathrm{nm}_A \quad} & (A \otimes A)^{\mathrm{C}_2} \\ & {\scriptstyle b_A} \searrow \quad \nearrow {\scriptstyle p_A} & \\ & \Gamma(A) & \end{array}$$

where $\mathrm{nm}_A\colon (A \otimes A)_{\mathrm{C}_2} \to (A \otimes A)^{\mathrm{C}_2}$ is the norm map of $A \otimes A$ considered with its C_2-action by flipping factors.

Proposition. *The two resulting sequences*

$$0 \longrightarrow \ker(\mathrm{nm}_A) \xrightarrow{b_A} \Gamma(A) \xrightarrow{p_A} (A \otimes A)^{\mathrm{C}_2} \longrightarrow 0$$

and

$$0 \longrightarrow (A \otimes A)_{\mathrm{C}_2} \xrightarrow{b_A} \Gamma(A) \xrightarrow{p_A} \mathrm{coker}(\mathrm{nm}_A) \longrightarrow 0$$

are exact.

This result in particular recovers the computations from [Whi50, Section 5], such as $\Gamma(\mathbb{Q}) = \mathbb{Q}$ or $\Gamma(P_l) = 0$, where P_l is the Prüfer group at the prime l, and also formally implies $\Gamma(\mathbb{Z}/2^k) = \mathbb{Z}/2^{k+1}$ once more. For the proof we shall need to know the cokernel of the norm map above.

Lemma. *The map*

$$\Delta_A\colon A/2 \longrightarrow \mathrm{coker}(\mathrm{nm}_A), \quad [a] \longmapsto [a \otimes a]$$

is an isomorphism for every abelian group A.

Proof. It is easy to check that Δ_A is a well-defined homomorphism for all abelian groups A and an isomorphism for $A = \mathbb{Z}/n$, $n \geq 0$, as in this case the norm map in question identifies with the multiplication by 2 on A itself. But by direct inspection both sides commute with direct sums (using the fact that the norm is an isomorphism on abelian groups with (co)induced C_2-action) and filtered colimits, so the class of groups for which Δ_A is an isomorphism is closed under these operations. But by the classification of finitely generated abelian groups, the smallest subcategory of Ab containing the quotients of $\mathbb{Z}$ that is closed under direct sums and filtered colimits is Ab itself. □

While there is a completely analogous natural map

$$A[2] \longrightarrow \ker(\mathrm{nm}_A), \quad a \longmapsto [a \otimes a],$$

which has abstractly isomorphic source and target for A finitely generated, it is not generally an isomorphism (e.g. for $\mathbb{Z}/2^k$, $k \geq 2$, it simply vanishes), and indeed

$$P_2[2] \cong \mathbb{Z}/2 \quad \text{whereas} \quad \ker(\mathrm{nm}_{\mathrm{P}_2}) = 0.$$

Instead we have:

Lemma. *The maps*

$$A[2^{k+1}] \longrightarrow \ker(\mathrm{nm}_A), \quad a \longmapsto 2^k[a \otimes a]$$

assemble to an isomorphism

$$\bigoplus_{k \geq 0} A[2^{k+1}]/(2A[2^{k+2}] + A[2^k]) \longrightarrow \ker(\mathrm{nm}_A)$$

for every abelian group A.

Proof. Again it is easy to check that the maps are well-defined homomorphisms. Note then that for $A = \mathbb{Z}/2^{k+1}$ only the kth summand on the left hand side is non-zero (namely it is $A/2$), and that this is indeed mapped isomorphically to $A[2] = \ker(\mathrm{nm}_A)$ via multiplication by 2^k in this case. The map in question is also trivially an isomorphism for all other quotients of $\mathbb{Z}$, so the same argument as in the previous lemma gives the claim. □

Proof of the Correction. Combined with the proposition above the preceeding lemma implies that the kernel of p_A is generally generated by the elements $2^{k+1}q(a) = 2^k b_A(a,a)$ with $a \in A[2^{k+1}]$, which yields the first claim.

For the second claim we have to show that $\ker(\mathrm{nm}_A) = 0$ if and only if every 2-power torsion element of A is divisible by 2. The backwards implication is immediate from the same lemma, and the forwards one follows by induction on the torsion order: Certainly, $A[0] = 2A[2]$ and by considering the kth summand on the left hand side, we find that $A[2^k] = 2A[2^{k+1}]$ implies

$$A[2^{k+1}] = 2A[2^{k+2}] + 2A[2^{k+1}] = 2A[2^{k+2}]$$

whenever $\ker(\mathrm{nm}_A) = 0$. □

It also follows that $(A \otimes A)^{\mathrm{C}_2}$ can be described in terms of generators $\Delta(a)$, $a \in A$, subject only to the relations

(1) $\Delta(a) = \Delta(-a)$
(2) $\Delta(a + b + c) + \Delta(a) + \Delta(b) + \Delta(c) = \Delta(a + b) + \Delta(a + c) + \Delta(b + c)$
(3) $2^k\Delta(a) = 0$ if $2^k a = 0$.

In our original statement we had missed the third of these relations (upon elision of which one obtains the presentation of $\Gamma(A)$ used by Whitehead). For consistency note that for any quadratic map $q\colon A \to B$ we have $\gcd(r,2)rq(a) = 0$ if $ra = 0$, so that the extra condition on the torsion elements is automatic for odd primes: From $r^2q(a) = q(ra)$ and $(r+1)^2q(a) = q(ra+a)$, one finds $r^2q(a) = 0 = 2rq(a)$ and this gives the claim.

Proof of the Proposition. The commutative diagram

$$\begin{array}{ccccc}
(A\otimes A)_{\mathrm{C}_2} & \xrightarrow{u\mapsto(0,u)} & W(A) & \xrightarrow{(a,u)\mapsto a} & A \\
\downarrow{\scriptstyle \mathrm{nm}_A} & & \downarrow{\scriptstyle p_A\circ \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}} \\
\mathrm{im}(\mathrm{nm}_A) & \longrightarrow & (A\otimes A)^{\mathrm{C}_2} & \longrightarrow & A/2
\end{array}$$

with exact rows shows surjectivity of p_A. Furthermore, via the snake lemma it induces the lower short exact sequence in

$$\begin{array}{ccccc}
A[2] & \longrightarrow & A & \xrightarrow{2} & 2A \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle a\mapsto(2a,-2[a\otimes a])} & & \downarrow{\scriptstyle \mathrm{id}} \\
\ker(\mathrm{nm}_A) & \longrightarrow & \ker(p_A\circ \mathrm{pr}) & \longrightarrow & 2A.
\end{array}$$

Another application of the snake lemma then gives an isomorphism

$$\ker(\mathrm{nm}_A) \longrightarrow \ker(p_A)$$

and thus the first claim. The second one follows from yet another application of the snake lemma to

$$\begin{array}{ccccc}
\ker(\mathrm{nm}_A) & \longrightarrow & (A\otimes A)_{\mathrm{C}_2} & \xrightarrow{\mathrm{nm}_A} & \mathrm{im}(\mathrm{nm}_A) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle b_A} & & \downarrow \\
\ker(\mathrm{nm}_A) & \xrightarrow{b_A} & \Gamma_A & \xrightarrow{p_A} & (A\otimes A)^{\mathrm{C}_2}.
\end{array}$$

□

Finally, we observe that one can immediately compute the cross-effect of Γ with the exact sequences above, which in particular implies that Γ is a quadratic functor (this is the starting point for the investigations in [EML54, Bau94]).

Corollary. *The quadratic form*

$$A\oplus B \longrightarrow A\otimes B, \quad (a,b)\longmapsto a\otimes b$$

induces an isomorphism

$$\Gamma(A\oplus B)\cong \Gamma(A)\oplus(A\otimes B)\oplus\Gamma(B)$$

for all abelian groups A and B.

Together with the computations given above, this determines $\Gamma(A)$ for all finitely generated abelian groups and since $A\mapsto\Gamma(A)$ commutes with filtered colimits by inspection, the same is in principle true for arbitrary abelian groups.

Proof. Consider the diagram

$$\begin{array}{ccccc} \ker(\mathrm{nm}_A) \oplus \ker(\mathrm{nm}_B) & \longrightarrow & \Gamma(A) \oplus \Gamma(B) & \longrightarrow & (A \otimes A)^{\mathrm{C}_2} \oplus (B \otimes B)^{\mathrm{C}_2} \\ \downarrow & & \downarrow & & \downarrow \\ \ker(\mathrm{nm}_{A \oplus B}) & \longrightarrow & \Gamma(A \oplus B) & \longrightarrow & ((A \oplus B) \otimes (A \oplus B))^{\mathrm{C}_2} \\ \downarrow & & \downarrow & & \downarrow \\ \ker(\mathrm{nm}_A) \oplus \ker(\mathrm{nm}_B) & \longrightarrow & \Gamma(A) \oplus \Gamma(B) & \longrightarrow & (A \otimes A)^{\mathrm{C}_2} \oplus (B \otimes B)^{\mathrm{C}_2} \end{array}$$

with horizontal exact sequences and vertical maps induced by the inclusions and projections. Since the left hand vertical maps are isomorphisms (e.g. by the second lemma), applying the snake lemma to the top two squares gives an isomorphism

$$\mathrm{coker}\Big[\Gamma(A) \oplus \Gamma(B) \to \Gamma(A \oplus B)\Big] \cong \mathrm{coker}\Big[(A \otimes A)^{\mathrm{C}_2} \oplus (B \otimes B)^{\mathrm{C}_2} \to ((A \oplus B) \otimes (A \oplus B))^{\mathrm{C}_2}\Big]$$

and the second cokernel is

$$((A \otimes B) \oplus (B \otimes A))^{\mathrm{C}_2} \cong A \otimes B.$$

via projection to the first summand. The middle vertical row displays this as a direct summand in $\Gamma(A \oplus B)$ and per construction the composite morphism

$$A \oplus B \xrightarrow{q_{A \oplus B}} \Gamma(A \oplus B) \xrightarrow{\mathrm{prop}_A} A \otimes B$$

is the one from the statement. □

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Mathematisches Institut, RFWU Bonn, Germany
E-mail address: `f.hebestreit@math.uni-bonn.de`

Department of Mathematical Sciences, University of Copenhagen, Denmark
E-mail address: `markus.land@math.ku.dk`

Mathematisches Institut, WWU Münster, Germany
E-mail address: `nikolaus@uni-muenster.de`