CONDENSED MATHEMATICS

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Abstract. These are lecture notes for my lecture "Condensed Mathematics" which I taught in the summer term 2022 at LMU Munich. The notes are unfinished and end where the term also ended. I intend to add more at some point. The notes are not thoroughly proof read - use at own risk. Comments are more than welcome.

Contents

1. Elementary category theory

In order to avoid talking about classes we will work in the following set theoretic setting. In addition to the usual ZFC axioms (Zermelo-Frankel set theory plus the axiom of choice) we will assume another axiom, called a large cardinal axiom:

Axiom . *For every cardinal κ there exists an inaccessible cardinal κ' with $\kappa' > \kappa$.*

A cardinal κ is called inaccessible if the (correctly to be defined) collection of sets $\mathcal{V}_{<\kappa}$ of cardinality less than κ itself satisfies the ZFC axioms. The collection $\mathcal{V}_{<\kappa}$ is called a universe. It turns out that this axiom cannot be proven from ZFC and in fact is logically independent. In particular, an inaccessible cardinal κ is larger than $\aleph_k$ for any k. From the axiom, we may fix a sequence

$$\kappa_0 < \kappa_1 < \kappa_2 < \ldots ...$$

of inaccessible cardinals and consider their associated universes $\mathcal{V}_{<\kappa}$.

1.1. **Definition** A set is called

(1) small, if it is contained in $\mathcal{V}_{<\kappa_0}$,
(2) large, if it is contained in $\mathcal{V}_{<\kappa_1}$,
(3) very large, if it is contained in $\mathcal{V}_{\kappa_2}$,
(4) very very large, if it is contained in $\mathcal{V}_{<\kappa_3}$, and so on.

In this lecture, we will not encounter sets that are not very large (I hope).

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1.2. **Example** The set of small sets is large. The set of large sets is very large, and so on.

1.3. **Definition** A category $\mathcal{C}$ consists of a (possibly large) set of objects $\mathrm{ob}(\mathcal{C})$, and for any two objects x and y a (also possibly large) set $\mathrm{Hom}_{\mathcal{C}}(x,y)$ of morphisms between them, equipped with composition maps

$$\mathrm{Hom}_{\mathcal{C}}(x,y) \times \mathrm{Hom}_{\mathcal{C}}(y,z) \to \mathrm{Hom}_{\mathcal{C}}(x,z)$$

and identities $* \to \mathrm{Hom}_{\mathcal{C}}(x,x)$ for all objects x, satisfying associativity and unitality. A morphism $f\colon c \to c'$ is called an isomorphism if there exists $g\colon c' \to c$ such that $gf = \mathrm{id}_c$ and $fd = \mathrm{id}_{c'}$. In this case we write f^{-1} for g (note that g is uniquely determined if its exists).

A category is called locally small if all hom sets are small, it is called small if it is locally small and the set of objects is also small. It is called essentially small if it is locally small and the set of isomorphism classes of objects are small.

1.4. **Remark** Usually in category theory a category would be defined as to have a (possibly proper) class of objects and for any two objects, a set of morphisms. In our language, this is what we call a locally small category. In general, we will not assume that a category is locally small.

1.5. **Definition** Let $\mathcal{C}$ be a category. Then $\mathcal{C}^{\mathrm{op}}$ denotes the category with $\mathrm{ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{ob}(\mathcal{C})$ and

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(c,c') = \mathrm{Hom}_{\mathcal{C}}(c',c)$$

with same identities and obvious composition structure.

1.6. **Definition** A functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of a map $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, $c \mapsto F(c)$, and for a pair of objects $c, c' \in \mathrm{ob}(\mathcal{C})$ a map $\mathrm{Hom}_{\mathcal{C}}(c,c') \to \mathrm{Hom}_{\mathcal{D}}(F(c),F(c'))$ which is

(1) compatible with identities: $F(\mathrm{id}_c) = \mathrm{id}_{F(c)} \in \mathrm{Hom}_{\mathcal{D}}(F(c),F(c))$, and
(2) compatible with composition: $F(g \circ f) = F(g) \circ F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(c),F(c''))$ for each pair of composable morphisms $c \xrightarrow{f} c' \xrightarrow{g} c''$.

A natural transformation between funtors $F,G\colon \mathcal{C} \to \mathcal{D}$ is a functor $\tau\colon \mathcal{C} \times [1] \to \mathcal{D}$ such that $\tau(-,0) = F$ and $\tau(-,1) = G$.

1.7. **Remark** Concretely, a natural transformation between F and G consists of a map $\tau_c\colon F(c) \to G(c)$ for all objects $c \in \mathrm{ob}(\mathcal{C})$ such that for each map $f\colon c \to c'$ the diagram

$$\begin{array}{ccc} F(c) & \xrightarrow{\tau_c} & G(c) \\ \big\downarrow{\scriptstyle F(f)} & & \big\downarrow{\scriptstyle G(f)} \\ F(c') & \xrightarrow{\tau_{c'}} & G(c') \end{array}$$

commutes.

1.8. **Definition** A natural transformation $\tau\colon F \to G$ is called a natural isomorphism if there exists a natural transformation $\tau'\colon G \to F$ such that $\tau \circ \tau' = \mathrm{id}_G$ and $\tau' \circ \tau = \mathrm{id}_F$. Equivalently, if each map τ_c is an isomorphism (Exercise).

1.9. **Definition** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called

(1) full, faithful, and fully faithful, respectively if for all pairs of objects $c, c' \in \mathrm{ob}(\mathcal{C})$ the induced map

$$\mathrm{Hom}_{\mathcal{C}}(c, c') \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(c), F(c'))$$

is surjective, injective, and bijective, respectively.

(2) essentially surjective, if for all $d \in \mathrm{ob}(\mathcal{D})$ there exists $c \in \mathrm{ob}(\mathcal{C})$ and an isomorphism $F(c) \cong d$.

(3) conservative, if F detects isomorphisms, that is, if f is an isomorphism if $F(f)$ is an isomorphism.

1.10. **Definition** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called an equivalence of categories if there exists $G\colon \mathcal{D} \to \mathcal{C}$ such that $FG \simeq \mathrm{id}_{\mathcal{D}}$ and $GF \simeq \mathrm{id}_{\mathcal{C}}$ (here, $\simeq$ refers to the existence of a natural isomorphism).

1.11. **Example** Sets, (abelian) groups, rings, vector spaces, moduls, topological spaces (see later), schemes, representations, spectra, derived categories.

1.12. **Definition** A (co)limit of a functor $F\colon I \to \mathcal{C}$ consists of an object of $\mathcal{C}$, written $\mathrm{colim}_I F$ equipped with maps $F(i) \to \mathrm{colim}_I F$ for every i which are compatible in the sense that for every morphism $i \to j$ in I, the diagram

$$\begin{tikzcd} F(i) \arrow[r] \arrow[d] & \mathrm{colim}_I F \\ F(j) \arrow[ur] & \end{tikzcd}$$

commutes (such a datum is also called a cone over F). This datum is required to satisfy the following universal property: Whenever given a further object $X \in \mathcal{C}$, also equipped with maps $F(i) \to X$ which are compatible in the above way, then there exists a unique morphism $\mathrm{colim}_I F \to X$ making the diagrams

$$\begin{tikzcd} F(i) \arrow[r] \arrow[d] & X \\ \mathrm{colim}_I F \arrow[ur] & \end{tikzcd}$$

commute (we then say that this is a colimit cone).

Dually, a limit of F is an object $\lim_I F$, equipped with maps $\lim_I F \to F(i)$, which are again compatible, satisfying the dual universal property: Whenever we are given an object X equipped with compatible morphisms $X \to F(i)$ for all $i \in I$, there exists a unique morphism $X \to \lim_I F$ making the obvious diagram commute.

1.13. **Remark** Notice that such a universal property specifies an object up to unique isomorphism. Notice also that the universal property refers to more than just the object $\mathrm{colim}_I F$. The reference maps are part of the data, and this is what makes the object unique up to unique isomorphism.

1.14. **Example** (1) A colimit of the empty diagram $\emptyset \to \mathcal{C}$ is an initial object: It is an object which admits a unique morphism to any other object. Dually, A limit of the

empty diagram $\emptyset \to \mathcal{C}$ is a terminal object: It is an object which admits a unique morphism from any other object.
(2) A colimit of the identity functor $\mathcal{C} \to \mathcal{C}$ is a terminal object.
(3) A colimit of the diagram $\bullet \leftarrow \bullet \to \bullet$ is called a pushout.
(4) A limit of the diagram $\bullet \to \bullet \leftarrow \bullet$ is called a pullback.

1.15. **Observation** One can phrase general (co)limits via inital and terminal objects. This point of view will be used later when we discuss limits and colimits in ∞-categories. Given a functor $F \colon I \to \mathcal{C}$ we can consider the category of (co)cones of this functor. Given a category I we consider a new category $I^{\triangleleft}$ and $I^{\triangleright}$, which are constructed from I by adding an initial respectively a terminal object. There is an obvious functor $I \to I^{\triangleleft}$ and $I \to I^{\triangleright}$. We can thus consider the functor categories

$$\mathrm{Fun}_F(I^{\triangleleft}, \mathcal{C}) \text{ and } \mathrm{Fun}_F(I^{\triangleright}, \mathcal{C})$$

of functors which restrict to F along the above mentioned inclusion. These are called the categories of cones and cocones over F, respectively. A colimit is then an initial cone and a limit is a terminal cocone.

1.16. **Lemma** *Let $\mathcal{C}$ be a category which admits coproducts and coequalizers. Then $\mathcal{C}$ is cocomplete. Dually, when $\mathcal{C}$ admits products and equalisers, it is complete.*

Proof. The second part follows from the first by considering $\mathcal{C}^{\mathrm{op}}$. Let $F \colon I \to \mathcal{C}$ be a diagram. Then

$$\mathrm{Coeq}\Big[\coprod_{(f\colon i \to j) \in \mathrm{Arr}(\mathcal{C})} F(i) \rightrightarrows \coprod_{k \in \mathrm{ob}(\mathcal{C})} F(k) \Big]$$

is a colimit as one checks by the universal property. Here, the maps are as follows: Restricted to the component $F(i)$ indexed over $f \colon i \to j$, the one map is the canonical inclusion $F(i) \to \coprod_{k \in \mathrm{ob}(\mathcal{C})} F(k)$ and the other map is the map $F(f) \colon F(i) \to F(j)$ followed by the canonical inclusion to $\coprod_{i \in \mathrm{ob}(\mathcal{C})} F(i)$. □

1.17. **Lemma** *The category* Set *is bicomplete.*

The following lemma is immediate from the definition of (co)limits, and the just established fact established that the category Set is bicomplete (else the statement does not make sense).

1.18. **Lemma** *Let $\mathcal{C}$ be a category and let $F \colon I \to \mathcal{C}$ be an I-shaped diagram in $\mathcal{C}$. Then, for every object $x \in \mathcal{C}$, there are canonical bijections*
(1) $\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_I F, x) \cong \lim_I \mathrm{Hom}_{\mathcal{C}}(F(i), x)$, *and*
(2) $\mathrm{Hom}_{\mathcal{C}}(x, \lim_I F) \cong \lim_I \mathrm{Hom}_{\mathcal{C}}(x, F(i))$.

Moreover, this property characterizes (co)limits uniquely.

1.19. **Definition** An adjunction consists of a pair of functors $(F \colon \mathcal{C} \to \mathcal{D}, G \colon \mathcal{D} \to \mathcal{C})$ together with a natural isomorphism between the two functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set}$ given by

$$\mathrm{Hom}_{\mathcal{D}}(F(-), -) \text{ and } \mathrm{Hom}_{\mathcal{C}}(-, G(-)).$$

1.20. **Remark** Equivalently, adjunctions can be described by unit and counit transformations satisfying the triangle equalities.

1.21. **Lemma** *Suppose given a functor $F\colon \mathcal{C} \to \mathcal{D}$. Specify for each object $d \in \mathrm{ob}(\mathcal{D})$ an object Gd together with a map $F(Gd) \to d$ such that the maps*

$$\mathrm{Hom}_{\mathcal{C}}(c, Gd) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(F(c), F(Gd)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(c), d)$$

are isomorphisms. Then the association $d \mapsto Gd$ assembles into a functor $G\colon \mathcal{D} \to \mathcal{C}$ which is right adjoint to F. There is an obvious dual statement for the existence of left adjoints.

1.22. **Lemma** *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor which admits right adjoints G and G'. Then there is a specified natural isomorphism between G and G'. (Adjoints, if they exist, are unique up to unique isomorphism).*

Proof. Consider the following two natural bijections

$$\mathrm{Hom}_{\mathcal{C}}(Gx, G'x) \cong \mathrm{Hom}_{\mathcal{D}}(FGx, x) \cong \mathrm{Hom}_{\mathcal{C}}(Gx, Gx).$$

Then the identity of Gx corresponds to a natural transformation $G \to G'$. Applying the same trick for $\mathrm{Hom}_{\mathcal{C}}(G'x, Gx)$ shows that this must be a natural isomorphism. $\square$

1.23. **Lemma** *For an adjunction $(F, G, \eta, \varepsilon)$ we have*

(1) *F is fully faithful if and only if the unit η is an isomorphism,*
(2) *G is fully faithful if and only if the counit ε is an isomorphism,*
(3) *F is an equivalence of categories if and only if η and ε are isomorphisms.*

Moreover, F is an equivalence (with inverse G) if F is fully faithful and G is conservative (and vice versa).

1.24. **Definition** A category is called (co)complete, if it admits (co)limits indexed over arbitrary small (co)limits. It is called bicomplete if it is both complete and cocomplete.

1.25. **Lemma** *If $\mathcal{C}$ is bicomplete, then (co)lim is left/right adjoint to the constant diagram functor. In particular, forming (co)limits determines a functor*

$$\mathrm{Fun}(I, \mathcal{C}) \to \mathcal{C}.$$

Proof. Let's spell out the colimit case. Consider the constant functor $\mathrm{const}\colon \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$. Now we specify, for each functor $F\colon I \to \mathcal{C}$ an object, namely $\mathrm{colim}_I F$. Part of the datum of a colimit are compatible maps $\{F(i) \to \mathrm{colim}_I F\}_{\{i \in I\}}$ which are easily seen to assemble into a natural transformation

$$F \to \mathrm{const}(\operatorname*{colim}_I F).$$

Then we consider the composite

$$\mathrm{Hom}_{\mathcal{C}}(\operatorname*{colim}_I F, X) \to \mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(\mathrm{const}(\operatorname*{colim}_I F), \mathrm{const}X) \to \mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(F, \mathrm{const}X)$$

which is a bijection by the universal property of a colimit. The lemma thus follows from Lemma 1.21. The case of limits is completely analogous. $\square$

1.26. **Lemma** *Given an adjunction with $F\colon \mathcal{C} \to \mathcal{D}$ being left adjoint to $G\colon \mathcal{D} \to \mathcal{C}$, and given a further auxiliary small category I, then the functors*

$$F_*\colon \mathrm{Fun}(I, \mathcal{C}) \rightleftarrows \mathrm{Fun}(I, \mathcal{D})\colon G_*$$

again form an adjoint pair (with F_* left adjoint to G_*).

Proof. The adjunction is determined by a counit map $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ and a unit map $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ that satisfy the trinagle identities. We now use these to construct counit and unit maps for the pair of functors (F_*, G_*) as follows: Let $\varphi \in \mathrm{Fun}(I, \mathcal{D})$. We need to specify a natural map $\varepsilon_*\colon F_*(G_*(\varphi)) \to \varphi$ of functors $I \to \mathcal{D}$, so let $x \in \mathcal{E}$. We define the new counit ε_* to be the map

$$F(G(\varphi(x))) \xrightarrow{\varepsilon_{\varphi(x)}} \varphi(x).$$

It is easy to see that this is natural in φ, since ε itself is a natural transformation. Similarly we define a natural transformation $\eta_*\colon \psi \to G_*F_*(\psi)$ to be given by

$$\psi(y) \xrightarrow{\eta_{\psi(y)}} G(F(\psi(y))).$$

It is then easy to see that the snake identities are satisfied, because (ε, η) satisfy the snake identities. □

1.27. **Proposition** *Let* $\mathcal{C}$ *be a bicomplete category, then* $\mathrm{Fun}(I, \mathcal{C})$ *is bicomplete as well. A (co)limit of a diagram* $X\colon J \to \mathrm{Fun}(I, \mathcal{C})$ *is given by the functor sending* $i \in I$ *to* $\mathrm{colim}_J X(j)(i)$*.*

Proof. Let us argue that $\mathrm{Fun}(I, \mathcal{C})$ is cocomplete. The completeness argument is similar (or can be formally deduced from this case by applying op correctly). We claim that the composite

$$\mathrm{Fun}(J, \mathrm{Fun}(I, \mathcal{C})) \cong \mathrm{Fun}(I, \mathrm{Fun}(J, \mathcal{C})) \xrightarrow{\mathrm{colim}_J} \mathrm{Fun}(I, \mathcal{C})$$

is a colimit functor we wish to show exists. By Lemma 1.26 this functor has a right given by

$$\mathrm{const}_*\colon \mathrm{Fun}(I, \mathcal{C}) \to \mathrm{Fun}(I, \mathrm{Fun}(J, \mathcal{C})) \cong \mathrm{Fun}(J, \mathrm{Fun}(I, \mathcal{C}))$$

it the proposition is shown once we convince ourselves that this is itself the constant functor (which is immediate from the definition), as then we allude to Lemma 1.25. □

1.28. **Definition** Let $\mathcal{C}$ be a category. We denote the category of functors $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$ by $\mathcal{P}(\mathcal{C})$ and call it the category of presheaves on $\mathcal{C}$. An object $x \in \mathcal{C}$ determines a *representable* presheaf, namely the presheaf $\mathrm{Hom}_{\mathcal{C}}(-, x)$ which sends $y \in \mathcal{C}$ to the set of morphisms from y to x. This determines a functor $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ which is called the *Yoneda embedding.*

1.29. **Lemma** *The Yoneda lemma: Let* $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ *be a functor and* $x \in \mathcal{C}$ *an object. Then the map*

$$\mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-, x), F) \to F(x)$$

given by sending η *to* $\eta(\mathrm{id}_x)$ *is a bijection. Moreover, the Yoneda embedding* $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ *is fully faithful.*

Proof. The inverse is given by sending an element $s \in F(x)$ to the function $\mathrm{Hom}_{\mathcal{C}}(y, x) \to F(y)$ sending f to $f^*(s)$. It is an explicit check to see that this is a natural transformation and an inverse the the above described map. The fully faithfulness follows immediately from the Yoneda Lemma: The effect of the Yoneda embedding on morphisms is the map

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-, x), \mathrm{Hom}_{\mathcal{C}}(-, y))$$

given by sending f to

$$\mathrm{Hom}_{\mathcal{C}}(z, x) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(z, y).$$

We claim that this map is inverse to the map described in the Yoneda lemma, which is given by sending a map $f\colon \mathrm{Hom}_{\mathcal{C}}(x, y)$ to the function $\mathrm{Hom}_{\mathcal{C}}(z, x) \to \mathrm{Hom}_{\mathcal{C}}(z, y)$ given by sending φ to $\varphi^*(f) = f_*\varphi$. □

1.30. **Lemma** *Left adjoints preserve colimits, right adjoints preserve limits.*

Proof. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor which admits a right adjoint, say G. Let $X\colon I \to \mathcal{C}$ be a diagram which has a colimit $\operatorname{colim}_I X \in \mathcal{C}$. We claim that F sends that colimit to a colimit of the diagram $I \to \mathcal{C} \to \mathcal{D}$. In formulas, we claim that the canonical map $\operatorname{colim}_I F(X(i)) \to F(\operatorname{colim}_I X(i))$ induced from the compatible maps $F(X(i)) \to F(\operatorname{colim}_I X(i))$ that are part of the datum of the colimit (and then applying F) is an isomorphism. To see this, it suffices to show that it induces a bijection on hom sets for all other objects $y \in \mathcal{D}$:

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{D}}(F(\operatorname*{colim}_I X(i)), y) &\cong \operatorname{Hom}_{\mathcal{C}}(\operatorname*{colim}_I X(i), Gy) \\
&\cong \lim_I \operatorname{Hom}_{\mathcal{C}}(X(i), Gy) \\
&\cong \lim_I \operatorname{Hom}_{\mathcal{D}}(F(X(i)), y) \\
&\cong \operatorname{Hom}_{\mathcal{D}}(\operatorname*{colim}_I F(X(i)), y)
\end{aligned}$$

so we are done by the Yoneda lemma, see Lemma 1.29. The argument for the claim that right adjoints preserve limits is similar. ☐

Es gibt noch diverse Dinge mehr über Kategorien zu sagen, für uns relevant werden die Begriffe von abelschen und (deren) derivierte Kategorien. Wir werden jetzt aber erstmal Kategorien Kategorien sein lassen und machen mit den grundlegenden Dingen in mengentheoretischer Topologie weiter, welche eine zentrale Rolle in dem Begriff einer verdichteten Menge sein werden.

2. Point-set topology

2.1. **Definition** (1) A topology on a set X consists of a collection $\mathcal{O}$ of subsets of X, called *open sets*, satisfying the following axioms:
   (a) $X, \varnothing \in \mathcal{O}$,
   (b) If $U_1, \dots, U_n \in \mathcal{O}$, then $U_1 \cap \dots \cap U_n \in \mathcal{O}$,
   (c) If I is a set and $U_i \in \mathcal{O}$ for all $i \in I$, then $\bigcup\limits_{i \in I} U_i \in \mathcal{O}$.

   A subset $A \subseteq X$ is called closed if its complement $X \setminus A$ is open. A subset $N \subseteq X$ is called a neighborhood of aA point $x \in X$, if N contains an open U which contains x.
(2) A subcollection $\mathcal{B}$ of a topology $\mathcal{O}$ is called a basis if every element in $\mathcal{O}$ is a union of elements in $\mathcal{B}$,
(3) A subcollection $\mathcal{S}$ of a topology $\mathcal{O}$ is called a subbasis, if every element in $\mathcal{O}$ is a union of finite intersections of elements in $\mathcal{S}$.
(4) A map $f\colon (X, \mathcal{O}) \to (X', \mathcal{O}')$ of topological spaces is called continuous if $f^{-1}(\mathcal{O}') \subseteq \mathcal{O}$, i.e. if preimages of open sets are open. Equivalently, if preimages of closed sets are closed.
(5) A map is called open if $f(\mathcal{O}) \subseteq \mathcal{O}'$, i.e. f maps open sets to open sets. It is called closed if f maps closed sets to closed sets. In general, there is no implication between the conditions *open* and *closed* for maps of topological spaces.

We let Top be the category of topological spaces with continuous maps as morphisms.

2.2. **Example** A metric space (M, d) gives rise to a topological space. A basis for the open sets on M are given by the open balls of radius ≤ 1 around arbitrary points:

$$\mathcal{B} = \{U_\epsilon(m) = \{x \in M \mid d(x, m) < \epsilon\} \quad | \quad m \in M, \epsilon < 1\}$$

This includes any closed subset of $\mathbb{R}^n$ and therefore manifolds are topological spaces (for a rather ill-defined notion of manifolds: The smooth submanifolds of $\mathbb{R}^n$ - In general, a smooth manifold is by definition a topological space equipped with extra structure and further properties).

2.3. **Definition** (1) Let $(X, \mathcal{O})$ be a topological space and $Y \subseteq X$ a subset. The subspace topology on Y consists of the collection suggestively written $Y \cap \mathcal{O}$ whose elements are given by $Y \cap U$ for some $U \in \mathcal{O}$.
(2) Let $(X, \mathcal{O})$ be a topological space and $p\colon X \to Y$ a surjection. The quotient topology on Y is given by those sets U such that $p^{-1}(U)$ is open.

2.4. **Lemma** *The forgetful functor* Top $\to$ Set *admits left and right adjoints. In particular it preserves all colimits and limits.*

Proof. We consider the functor sending a set S to the same set equipped with the indiscrete topology, denoted $S^{i\delta}$ consisting only of $\varnothing$ and S. Likewise we consider the set S equipped with the discrete topology, denoted S^δ, where every subset of S is open. We then note that

$$\mathrm{Hom}_{\mathrm{Top}}(X^\delta, Y) = \mathrm{Hom}_{\mathrm{Set}}(X, Y)$$

and that

$$\mathrm{Hom}_{\mathrm{Top}}(Y, X^{i\delta}) = \mathrm{Hom}_{\mathrm{Set}}(Y, X)$$

and that these isomorphisms are natural in both X and Y. Therefore, we have found left and right adjoints. $\square$

Exercise. Is the forgetful functor Top $\to$ Set conservative?

2.5. **Lemma** *The category* Top *of topological spaces is cocomplete and complete.*

Proof. Since the forgetful functor Top $\to$ Set has left and right adjoints, we know what the underlying set of the limit and colimit of diagrams of spaces must be. We first define topologies on the disjoint union and the product of sets to obtain coproducts and products in Top. The topology on a product of topological spaces X_i has subbasis given by the sets of the form $p_i^{-1}(U_i)$, where $p_i\colon \prod_{k\in I} X_k \to X_i$ is the projection and $U_i \subseteq X_i$ is open. The topology on a disjoint union of topological spaces X_i has as subbasis the sets of the form $\iota_i(U_i) \subseteq \coprod_{i\in I} X_i$. We then topologise a general limit with the subspace topology of the product, and a general colimit with the quotient topology from the disjoint union.

Now, given a general diagram $F\colon I \to$ Top, we know that the underlying set of a limit must be given by $\lim_I F$ performed in the category of sets (i.e. discarding the topology on the sets $F(i)$). We have argued earlier that this is a subset of $\prod_{i\in I} F(i)$ and so we simply give $\lim_I F$ the subspace topology according to Definition 2.3. One then checks that the resulting topological space is indeed a limit.

For colimits one argues similarly: We know that $\operatorname{colim}_I F$ receives a surjection from $\coprod_{i\in I} F(i)$, and so we give it the quotient topology. Again, one checks that this makes the resulting topological space a colimit. □

2.1. **Connectedness.**

2.6. **Definition** A non-empty topological space X is called

(1) connected, if the following holds: The only non-empty open and closed subset of X is X itself.
(2) locally connected, if for all $x \in X$ and all open sets $U \ni x$ containing x, there is a connected and open subspace $x \in V \subseteq U$.

A connected component of X at a point $x \in X$ is a maximal connected subspace containing x.

2.7. **Remark** Put differently: A topological space is called connected if it has exactly one connected component. Indeed, we recall that a connected components are determined by an equivalence relation on the set underlying the space X (namely $x \sim x'$ if they lie in the same connected component) and therefore the number of connected components is the cardinality of a quotient of X by an equivalence relation. We conclude that the empty set has 0 connected components and is therefore not connected.

2.8. **Lemma** *The union of all connected subspaces containing x is connected. It is therefore the (unique) connected component containing x, we write $C(x)$ for it.*

Proof. Let A be a non-empty open and closed subset of the union T of all connected subspaces C containing x. As A is non-empty, there is such C with $C \cap A \neq \varnothing$ and this intersection is a closed and open subset of C. Since C is connected, we find $C \cap A = C$ and therefore that $C \subseteq A$. In particular, we have that $x \in A$. We conclude that the intersection $C' \cap A$ of *every* connected subspace C' containing x is non-empty (as it contains x) and therefore, by the same reasoning as before, $C' \subseteq A$. We conclude that $T \subseteq A$. Moreover, by definition $A \subseteq T$ and therefore $A = T$. We conclude that T is connected. ☐

2.9. **Lemma** *Let $f\colon X \to Y$ be a continuous map and $A \subseteq X$ a connected subspace. Then $f(A) \subseteq Y$ is also connected.*

Proof. Let $B \subseteq f(A)$ be a non-empty, open and closed subspace of $f(A)$. Recall that $f\colon A \to f(A)$ is continuous (as follows from the definition of the subspace topology). Therefore $f^{-1}(B)$ is open and closed in A, and it is non-empty since B is non-empty. Hence, $f^{-1}(B) = A$ since A is connected. But then $B = f(f^{-1}(B)) = f(A)$ and therefore, $f(A)$ is connected. ☐

2.10. **Definition** Let $U \subseteq X$ be a subspace. The closure of U, denoted $\overline{U}$, is given by

$$\overline{U} = \bigcap_{U \subseteq A \bar{\subseteq} X} A$$

where A runs through the closed subspaces of X which contain U.

2.11. **Lemma** *Let $C \subseteq X$ be a connected subspace. Then $\overline{C}$ is also connected.*

Proof. First we note the following: Let $U \subseteq X$ be an open subspace and $C \subseteq X$ any subspace. If $U \cap C = \varnothing$, then $U \cap \overline{C} = \varnothing$ as well. Indeed, the assumption implies that $C \subseteq X \setminus U$ and $X \setminus U$ is closed. The definition of $\overline{C}$ then implies that $\overline{C} \subseteq X \setminus U$ as well. Therefore $U \cap \overline{C} = \varnothing$ as claimed.

Now let C be connected and $A \subseteq \overline{C}$ non-empty, open and closed. Then $A \cap C$ is open and closed in C and non-empty by what we just argued. We deduce that $A \cap C = C$ since C is connected, and therefore that $C \subseteq A$. By definition of $\overline{C}$ we then find that $\overline{C} \subseteq A$, since A is closed. Therefore $A = \overline{C}$ and hence $\overline{C}$ is connected. ☐

2.12. **Lemma** *Let X be a space.*

(1) *Connected components of X are closed.*
(2) *Connected components of X are open if X is locally connected.*

Proof. (1) Let C be a connected component of X. Then $C \subseteq \overline{C}$ and $\overline{C}$ is again connected by Lemma 2.11. Since C is a maximal connected subspace, $C = \overline{C}$ and is therefore closed. ☐

Exercise. Prove statement (2) of Lemma 2.12. Note also that an open subspace of a locally connected space is again locally connected. Statement (2) can then be upgraded to the statement that X is locally connected if and only if all components of all open subspaces are open. Find an example where connected components are not open.

Exercise. Recall what path connected spaces are. Discuss the relation between the notions (locall) path connected and (locally) connected.

The drastic opposite of connected spaces (those where $C(x) = X$ for all $x \in X$) are the totally disconnected spaces:

2.13. **Definition** A topological space X is called totally disconnected if for all $x \in X$, we have $C(x) = \{x\}$, or equivalently, if every connected subspace of X is of the form $\{x\}$ for some $x \in X$.

2.2. Compactness.

2.14. **Definition** A topological space X is called compact, if for every open cover $X = \bigcup_{i\in I} U_i$ there exists a finite subcover, that is, a finite subset $J \subseteq I$ such that $X = \bigcup_{j\in J} U_j$.

2.15. **Lemma** (1) *Let $f\colon X \to Y$ be a continuous map and $K \subseteq X$ a compact subspace. Then $f(K) \subseteq Y$ is compact.*
(2) *Closed subspaces of compact spaces are compact.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of $f(K)$. Since $f\colon K \to f(K)$ is continuous, $\{f^{-1}(U_i)\}_{i\in I}$ forms an open cover of K. Since K is compact, there exists a finite subset $J \subseteq I$ so that $K = \bigcup_{j\in J} f^{-1}(U_j)$. Then we find $f(K) \subseteq f(\bigcup_{j\in J} f^{-1}(U_j)) = \bigcup_{j\in J} U_j$ proving (1). To see (2), let $A \subseteq X$ be a closed subspace of a compact space X. Let $\{U_i\}_{i\in I}$ be an open cover of A. Then $\{U_i\}_{i\in I} \cup \{X \setminus A\}$ is an open cover of X. There is then a finite subcover given by a finite subset $J \in I$ and we find that $\{U_j\}_{j\in J}$ is a finite subcover, showing that A is compact. □

The following is an equivalent formulation of compactness which we will use later.

2.16. **Lemma** *Let X be a compact space, $\{Z_i\}_{i\in I}$ a collection of closed subspaces such that each intersection of finitely many Z_i's is non-empty. Then the intersection of all Z_i's is non-empty.*

Proof. We prove the contraposition. Let us therefore assume that
$$\bigcap_{i\in I} Z_i = \varnothing.$$
Let $U_i = X \setminus Z_i$ be the open complements of the closed sets Z_i. We then obtain the open cover
$$X = \bigcup_{i\in I} U_i.$$
Since X is compact, there exists a finite subset $J \subseteq I$ so that $X = \bigcup_{j\in J} U_j$. This implies that $\bigcap_{j\in J} Z_j = \varnothing$, as needed. □

An important theorem about compact spaces is Tychonoff's theorem.

2.17. **Theorem** *An arbitrary product of compact spaces X_i is compact. If all X_i are non-empty, the converse holds as well.*

To prove this, we will make use of the notion of (ultra)filters. These will also appear later again in the context of the Stone-Cech compactification and are in any case an object worth introducing.

2.18. **Definition**
A filter $\mathcal{F}$ on a set X is a collection of subsets of X satisfying the following axioms:
(1) $X \in \mathcal{F}$ und $\varnothing \notin \mathcal{F}$,
(2) if $U \in \mathcal{F}$ and $U \subseteq V$, then $V \in \mathcal{F}$.
(3) if $U_1, \ldots, U_n \in \mathcal{F}$, then $U = \bigcap U_i \in \mathcal{F}$.

We write $\mathcal{F} \subseteq \mathcal{F}'$ if $U \in \mathcal{F}$ implies that $U \in \mathcal{F}'$. An ultrafilter is a maximal filter with respect to $\subseteq$.

2.19. **Lemma** *Let $\mathcal{A}$ be a collection of subsets of a set X such that each finite subcollection has non-empty intersection. Then*

$$\langle \mathcal{A} \rangle = \{ B \subseteq X \quad | \quad \exists A_1, \ldots, A_n \in \mathcal{A}, A_1 \cap \cdots \cap A_n \subseteq B \}$$

is a filter (the smallest filter containing $\mathcal{A}$). We call $\langle \mathcal{A} \rangle$ the filter generated by $\mathcal{A}$.

Proof. The axioms of a filter are immediate. It is also clear that $\langle \mathcal{A} \rangle$ is the smallest filter containing $\mathcal{A}$. □

2.20. **Remark** If $\mathcal{A}$ is a collection of subsets such that there is a finite subcollection with empty intersection, then $\mathcal{A}$ is not contained in any filter (else that filter would contain the empty-set). The assumption in Lemma 2.19 can therefore not be dropped.

2.21. **Lemma** *Let X be a set.*

(1) *A filter $\mathcal{F}$ on X is an ultrafilter if and only if for all $A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.*
(2) *For an ultrafilter $\mathcal{F}$ we have the following. If $A \cup B \in \mathcal{F}$, then $A \in F$ or $B \in \mathcal{F}$.*

Proof. (1) Let $\mathcal{F}$ be an ultrafilter and let $A \subseteq X$ so that $X \setminus A \notin \mathcal{F}$. We aim to show that $A \in \mathcal{F}$. Now, for $B \in \mathcal{F}$ we find that B is not a subset of $X \setminus A$ (else $X \setminus A \in \mathcal{F}$). Hence, $B \cap A \neq \varnothing$ for all $B \in \mathcal{F}$. Consequently, the collection of subset $\mathcal{F} \cup \{A\}$ has the property that each finite subcollection has non-empty intersection. By Lemma 2.19, it generates a filter $\mathcal{F}'$ with $\mathcal{F} \subseteq \mathcal{F}'$. Since $\mathcal{F}$ is an ultrafilter, we find $\mathcal{F} = \mathcal{F}'$ and therefore that $A \in \mathcal{F}$ (since $A \in \mathcal{F}'$).

Conversely, let us assume that $\mathcal{F}$ is a filter such that for all $A \subseteq X$, either A or $X \setminus A$ is contained in $\mathcal{F}$. Let $\mathcal{F} \subseteq \mathcal{G}$ for some filter $\mathcal{G}$ and $A \in \mathcal{G}$. We deduce that $X \setminus A \notin \mathcal{G}$ and therefore also that $X \setminus A \notin \mathcal{F}$. Hence $A \in \mathcal{F}$ and therefore $\mathcal{F} = \mathcal{G}$. Consequently, $\mathcal{F}$ is an ultrafilter.

(2) Let $A \cup B \in \mathcal{F}$ and $A \subseteq X$. If $A \in \mathcal{F}$ we are done. If $A \notin \mathcal{F}$, then $X \setminus A \in \mathcal{F}$ since $\mathcal{F}$ is an ultrafilter. We also have

$$B \setminus A = X \setminus A \cap (A \cup B).$$

Therefore, $B \setminus A \in \mathcal{F}$. Since $B \setminus A \subseteq B$, we find that $B \in \mathcal{F}$. □

2.22. **Lemma** *Every filter $\mathcal{F}$ is contained in an ultrafilter $\bar{\mathcal{F}}$.*

Proof. This follows from Zorn's Lemma, as soon as we show that the filtered colimit $\mathcal{F} = \mathrm{colim}_{i \in I} \mathcal{F}_i$ of filters $\mathcal{F}_i$, indexed over a totally ordered set I, with injective transition maps, i.e. where for $i \leq i'$ the map $\mathcal{F}_i \to \mathcal{F}'_i$ is an inclusion of filters, is again a filter. This, however, is again immediate from the definitions. □

2.23. **Definition** Let $f \colon X \to Y$ a map of sets and $\mathcal{G}$ a collection of subsets of X. We denote by $f_*(\mathcal{G})$ the collection of subsets $B \subseteq Y$ such that $f^{-1}(B) \in \mathcal{G}$.

2.24. **Lemma** *Let $f \colon X \to Y$ be a map of sets and $\mathcal{F}$ an (ultra)filter on X. Then $f_*(\mathcal{F})$ is an (ultra)filter on Y.*

Proof. From $f^{-1}(Y) = X$ and $f^{-1}(\varnothing) = \varnothing$ we get axiom (1). Now let $B \in f_*(\mathcal{F})$ and $B \subseteq B'$. Then $f^{-1}(B) \subseteq f^{-1}(B')$ so that $f^{-1}(B') \in \mathcal{F}$ and therefore $B' \in f_*(\mathcal{F})$, giving axiom (2). Likewise, for $B_1, \ldots, B_n \in f_*(\mathcal{F})$. Since $f^{-1}(\bigcap_i B_i) = \bigcap_i f^{-1}(B_i)$, we find that $\bigcap_i B_i \in f_*(\mathcal{F})$, giving axiom (3). Assume now that $\mathcal{F}$ is in addition an ultrafilter. By Lemma 2.21 it suffices to show that for $B \subseteq Y$, we have $B \in f_*(\mathcal{F})$ or $Y \setminus B \in f_*(\mathcal{F})$. By definition this means that $f^{-1}(B) \in \mathcal{F}$ or $f^{-1}(Y \setminus B) \in \mathcal{F}$ which is true since $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ and $\mathcal{F}$ is an ultrafilter. □

2.25. **Definition** Let X be a topological space and $x \in X$ a point. The neighborhood filter $\mathcal{U}(x)$ consists of the neighborhoods of x, that is all subsets $N \subseteq X$ with $x \in N$ and such that there exists an open $U \subseteq N$ with $x \in U$ - this is also the filter generated by the collection $\mathcal{A}$ of open subsets containing x. We say that a Filter $\mathcal{F}$ converges to a point x if $\mathcal{U}(x) \subseteq \mathcal{F}$. We then also say that x is a limit point of $\mathcal{F}$ and that $\mathcal{F}$ is convergent.

2.26. **Theorem** *A topological space is compact if and only if every ultrafilter has at least one limit point.*

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X and assume that every ultrafilter on X has at least one limit point. Consider the set $\mathcal{A} = \{A_i\}_{i \in I}$ with $A_i = X \setminus U_i$. If there is no finite subcover of $\mathcal{U}$, then the collection $\mathcal{A}$ has the property that every finite intersection of elements in $\mathcal{A}$ is non-empty by the argument used in the proof of Lemma 2.16. Consequently, $\mathcal{A}$ is contained in an ultrafilter $\mathcal{F}$, by Lemma 2.19 and Lemma 2.22. $\mathcal{F}$ then has a limit point x for which we can find $i \in I$ such that $x \in U_i$. Since $\mathcal{F}$ converges to x, we have $U_i \in \mathcal{F}$. However, by construction also $A_i \in \mathcal{F}$ which is a contradiction, so a finite subcover of $\mathcal{U}$ must exist, hence X is compact.

Conversely, let $\mathcal{F}$ be an ultrafilter on X which has no limit point. For $x \in X$ consider an open U_x which is not contained in $\mathcal{F}$. Then $\mathcal{U} = \{U_x\}_{x \in X}$ is an open cover of X. Suppose there is a finite subcover indexed by $x_1, \ldots, x_n$, then we obtain $U_{x_1} \cup \cdots \cup U_{x_n} = X \in \mathcal{F}$. By Lemma 2.21 part (2) we deduce that there is an $i \in \{1, \ldots, n\}$ such that $U_{x_i} \in \mathcal{F}$, again a contradiction. Therefore, there is no finite subcover of $\mathcal{U}$ and consequently, X is not compact. $\square$

Exercise. Let X be a topological space and $A \subseteq X$ a subset. Show that a point x lies in $\overline{A}$ if and only if there is a Filter $\mathcal{F}$ on A which converges to x in the sense that $\mathcal{U}(x) \cap A \subseteq \mathcal{F}$.

We now come to the proof of Tychonoff's theorem.

Beweis von Theorem 2.17. Let $\{X_i\}_{i \in I}$ be a collection of compact spaces. We wish to show that $\prod_I X_i$ is compact. So let $\mathcal{F}$ be an ultrafilter in $\prod_I X_i$ and let $p_j \colon \prod_I X_i \to X_j$ be the projection. By Lemma 2.24, $(p_j)_*(\mathcal{F})$ is an ultrafilter on X_j, so there is a limit point $x_j \in X_j$ for $(p_j)_*(\mathcal{F})$ by Theorem 2.26. We claim that the sequence $x = (x_j)_{j \in J}$ is a limit point of $\mathcal{F}$. For this, it suffices to show that any element U of a subbasis of the topology on $\prod_I X_i$ with $x \in U$ is contained in $\mathcal{F}$. A subbasis is given by $p_j^{-1}(U_j)$ for $j \in I$ and $U_j \subseteq X_j$ open. So we need to show that $p_j^{-1}(U_j) \in \mathcal{F}$ which is the case if and only if $U_j \in (p_j)_*(\mathcal{F})$ by definition of $(p_j)_*(\mathcal{F})$. This is the case by the assumption that x_j is a limit point of $(p_j)_*(\mathcal{F})$. $\square$

2.3. Separation axioms and Hausdorff spaces.

2.27. **Definition** A topological space is called

(1) Hausdorff (T2), if for all $x \neq x' \in X$ there are disjoint open sets U, U' with $x \in U$ and $x' \in U'$.
(2) regular (T3), if for all closed subsets $A \subseteq X$ and $x \in X \setminus A$, there are disjoint open sets U, V with $A \subseteq U$ and $x \in V$.
(3) normal (T4), if for all $A, A' \subseteq X$ disjoint closed, there are disjoint open sets U, U' with $A \subseteq U$ and $A' \subseteq U'$.

2.28. **Remark** Sometimes, it is required that normal and regular spaces are Hausdorff.

2.29. **Theorem** *A topological space is Hausdorff if and only if every Filter has at most one limit point.*

Proof. Let x and y be limit points of a filter $\mathcal{F}$ on X. By definition, every open set containing x or y is contained in $\mathcal{F}$. Since $\mathcal{F}$ is a filter, no two such sets can be disjoint. This implies the theorem. □

2.30. **Corollary** *A topological space is compact Hausdorff if and only if every Ultrafilter converges to precisely one limit point.*

Proof. The only if is immediate from Theorem 2.26 and Theorem 2.29. For the "if" part, it remains to show that any filter converges to at most one limit point. However, any limit point of a filter is also a limit point of any ultrafilter it is contained in. So there is at most one limit point for any filter. □

2.31. **Lemma** *Let X be a Hausdorff space.*

(1) *For $K \subseteq X$ compact and $y \in X \setminus K$, there are disjoint open sets $U \supseteq K$ and $V \ni y$.*
(2) *Compact subsets of X are closed.*

In particular, points are closed in X.

Proof. Let $K \subseteq X$ be compact. Since X is Hausdorff, for $y \in X \setminus K$ and $x \in K$ we can find disjoint open sets $U_x \ni x$ and $V_{y,x} \ni y$. Then $\{U_x\}_{x \in K}$ is an open cover of K. Since K is compact we can find a finite subcover, indexed by $x_1, \dots, x_n$. Set $V = V_{y,x_1} \cap \dots \cap V_{y,x_n}$ and $U = U_{x_1} \cup \dots \cup U_{x_n}$. Then $U \cap V = \varnothing$, $K \subseteq U$ and $y \in V$, showing (1).

For (2), let $K \subseteq X$ be compact. For $y \notin K$ we find $V_y \ni y$ and $U \subseteq K$ with U, V open and $U \cap V = \varnothing$, in particular $V_y \cap K = \emptyset$. But then we have $X \setminus K = \bigcup_{y \in X \setminus K} V_y$ so that K is closed.

Clearly, points are compact and hence closed by (2). □

2.32. **Lemma** *Compact Hausdorff spaces are normal.*

Proof. Let $A, A' \subseteq X$ be disjoint closed subsets and let $x \in A$. Since A' is closed and X is compact, A' is also compact by Lemma 2.15. By part (1) of Lemma 2.31 there are disjoint opens $V_x \supseteq A'$ and $U_x \ni x$. We then get that $A \supseteq \bigcup_{x \in A} U_x$, and since A is also compact, there is again a finite subcover indexed by $x_1, \dots, x_n \in A$, so we have $A \subseteq U_{x_1} \cup \dots \cup U_{x_n} = U$. Defining similarly as before $V = V_{x_1} \cap \dots \cap V_{x_n}$ we find that $A' \subseteq V$ and $U \cap V = \varnothing$. □

2.33. **Definition** Let $\mathrm{CH} \subseteq \mathrm{Top}$ be the full subcategory on compact Hausdorff spaces.

2.34. **Lemma** *Let $f\colon X \to Y$ be a continuous map with X compact and Y Hausdorff.*

(1) *The map f is closed.*
(2) *If f is surjective, it is a quotient map.*
(3) *If f is bijective, it is a homeomorphism.*

In other words, (3) implies that the forgetful functor CH $\to$ Set *is conservative.*

Proof. (1) Let $A \subseteq X$ be closed. By Lemma 2.15 part (2), A is compact. Hence by Lemma 2.15 part (1), $f(A)$ is also compact, and hence by Lemma 2.31, $f(A)$ is closed. (2) We recall that f is a quotient map if a subset $U \in Y$ is open if and only if $f^{-1}(U) \subseteq X$ is open. One implication follows from the continuity of f. For the other, assume $U \subseteq Y$ is such that $f^{-1}(U) \subseteq X$ is open. Then $X \setminus f^{-1}(U)$ is closed, and hence by (1) $f(X) \setminus U = f(X \setminus f^{-1}(U))$ is also closed. Since f is surjective, $X \setminus U$ is closed, therefore $U \subseteq Y$ is open as needed. (3) For a bijection $f\colon X \to Y$, the map f is closed (or open) if and only if the (set-theoretically defined) map $f^{-1}\colon Y \to X$ is continuous. $\square$

2.35. **Corollary** *Let $X \to Y$ be a continuous surjection of compact Hausdorff spaces. Then f is a quotient map, i.e. $U \subseteq Y$ is open if and only if $f^{-1}(U) \subseteq X$ is open. In other words, a map $g\colon Y \to Y'$ is continuous if and only if the composite gf is continuous.*

Proof. A surjection is a quotient map if and only if it is an open and continuous map. This was verified in Lemma 2.34. The "in other words" follows immediately. $\square$

2.36. **Lemma** *Let $X \in$* CH *and $x \in X$. The connected component $C(x)$ of x is the intersection of all open and closed subsets of X containing x.*

Proof. Let A be an open and closed subspace containing x. Then $A \cap C(x)$ is non-empty and an open and closed subspace of $C(x)$. Since $C(x)$ is connected, we deduce that $C(x) \subseteq A$, and therefore that $C(x)$ is contained in the intersection S of all open and closed subspaces containing x. We then need to show the other inclusion for which it suffices to show that S is connected. Let us write $S = B \amalg C$ for disjoint open and closed subspaces of S, and we wish to show that $S = B$ or $S = C$. Since S is closed in X, we also find that B and C are closed subspaces of X. By Lemma 2.32 X is normal, so we find disjoint open subspaces (of X) $U \supseteq B$ and $V \supseteq C$. We then have $S \subseteq U \cup V$, so $[X \setminus (U \cup V)] \cap S = \varnothing$. From the definition of S as an intersection of closed sets and Lemma 2.16 it follows that there exists an open and closed A containing x such that $[X \setminus (U \cup V)] \cap A = \varnothing$, or in other words, such that $A \subseteq U \cup V$. Since $U \cap V = \varnothing$, we find a that $A = (A \cap U) \amalg A \cap V$ is a disjoint decomposition into open and subsets of A and therefore $A \cap U$ and $A \cap V$ are open and closed in X. Assuming that x lies in $B \subseteq U$, we find that $S \subseteq A \cap U$ (since $A \cap U$ is open and closed in X and contains x). Consequently, we find that $S \cap (A \cap V) = \varnothing$, which implies that $S \cap C = \varnothing$ showing that $S = B$ and $C = \varnothing$. $\square$

Finally, we will need the following extension theorem of Tietze, which is itself an extension of Urysohn's lemma.

2.37. **Theorem** *Let $A \subseteq X$ be a closed subset of a normal space X and let $f\colon A \to \mathbb{R}$ be a continuous map. Then there exists a continuous map $F\colon X \to \mathbb{R}$ with $F_{|A} = f$.*

To prove this theorem we will need preliminary lemmata. We recall that a subspace $A \subseteq X$ is called dense if $\overline{A} = X$.

2.38. **Lemma** *Let $D \subseteq [0,1]$ be a dense subspace and let $(X, \mathcal{O})$ be a topological space. Let $\Phi\colon D \to \mathcal{O}$ be a map of posets such that for $d < d'$, we have $\overline{\Phi(d)} \subseteq \Phi(d')$. Then the function $f\colon X \to [0,1]$ given by*

$$f(x) = \inf\{d \in D \mid x \in \Phi(d)\}$$

is continuous. For this definition, we set the infimum over the empty set to be 1.

Proof. The sets $[0,a[$ and $]a,1]$ form a subbasis of the topology on $[0,1]$. But

$$f^{-1}([0,a[) = \{x \in X \mid f(x) < a\} = \bigcup_{d<a} \Phi(d) \text{ and}$$

$$f^{-1}(]a,1]) = \{x \in X \mid f(x) > a\} = \bigcup_{d>a} X \setminus \Phi(d) = \bigcup_{d>a} X \setminus \overline{\Phi(d)}.$$

Here the final equality holds because D is dense in $[0,1]$ and we have $\overline{\Phi(d)} \subseteq \Phi(d')$ for $d < d'$. □

The following is Urysohn's lemma.

2.39. **Theorem** *Let X be a normal space and A and B two disjoint closed subspaces. Then there is a continuous map $f\colon X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Proof. We will use the previous lemma for D the set of dyadic numbers $D = \{\frac{m}{2^n} \mid 0 \leq m \leq 2^n\}$. We construct the function $\Phi\colon D \to \mathcal{O}$ as follows. We first set $\Phi(1) = X \setminus B$. We now inductively define Φ in general. To start, choose disjoint open sets $\Phi(0) \supseteq A$ and $V \supseteq B$. Then $\overline{\Phi(0)} \cap V = \varnothing$, i.e. $\overline{\Phi(0)} \subseteq X \setminus B$. We have constructed

$$A \subseteq \Phi(0) \subseteq \overline{\Phi(0)} \subseteq \Phi(1) = X \setminus B.$$

Performing the same construction for A replaced by $\overline{\Phi(0)}$ gives an open set $\Phi(1/2)$ with

$$\overline{\Phi(0)} \subseteq \Phi(1/2) \subseteq \overline{\Phi(1/2)} \subseteq \Phi(1).$$

Performing the same construction for the pair $\overline{\Phi(1/2)}$ and B gives $\Phi(3/4)$, while performing the same construction for the pair $\overline{\Phi(0)}$ and $X \setminus \Phi(1/2)$ gives $\Phi(1/4)$. Inductively, this defines Φ as needed. Then we see that the map $f\colon X \to [0,1]$ obtained by Lemma 2.38 sends A to 0 and B to 1. □

2.40. **Proposition** *Let X be a normal space and $A \subseteq X$ a closed subspace. Every continuous map $f\colon A \to [0,1]$ has a continuous extension to a map $F\colon X \to [0,1]$.*

Proof. Let $0 < \epsilon \leq 1$. We say that a map $g_\epsilon\colon X \to [0,1]$ is an ϵ-extension of f if

(1) $|g(x)| \leq 1 - \epsilon$ for all $x \in X$, and
(2) $|g(a) - f(a)| \leq \epsilon$ for all $a \in A$.

For $\epsilon = 1$, the function $X \to [0,1]$ which is constant at 0 is such an extension. Now, given such an extension g_ϵ we construct and improvement of g_ϵ as follows. Consider the sets

$$C = \{a \in A \mid f(a) - g(a) \geq \epsilon/3\} \text{ and } D = \{a \in A \mid f(a) - g(a) \leq -\epsilon/3\}$$

By Theorem 2.39 we find a continuous map $u\colon X \to [-\epsilon/3, \epsilon/3]$ taking C to $-\epsilon/3$ and D to $\epsilon/3$. We consider then the map $g_\epsilon - u$. It satisfies

(a) $|(g_\epsilon - u)(x)| \leq 1 - \frac{2\epsilon}{3}$ for all $x \in X$,
(b) $|g_\epsilon(x) - (g_\epsilon - u)(x)| \leq \frac{\epsilon}{3}$, for all $x \in X$.
(c) $|f(a) - (g_\epsilon - u)(a)| \leq \frac{2\epsilon}{3}$, for all $a \in A$, and

Indeed, (a) and (b) are true by construction, and (c) can be checked on C, D and the complement of $C \cup D$ individually. Starting with $g^0 = 0$ and applying the above construction we find functions g^n which turn out to be $\epsilon = (\frac{2}{3})^n$-extensions of f. In addition, for $m, n \geq k$, we find that

$$|g^n(x) - g^m(x)| \leq (\tfrac{2}{3})^k$$

so the sequence g^n converges uniformly to a continuous map $g^\infty \colon X \to [0,1]$ which satisfies the required properties. □

Proof of Theorem 2.37. Choose a homeomorphism $\mathbb{R} \cong]-1,1[$. Since the inclusion $]-1,1[\subseteq [-1,1]$ is continuous, $f \colon A \to \mathbb{R}$ gives rise to a continuous map $f \colon A \to [-1,1]$. Let $\bar{F} \colon X \to [-1,1]$ be a continuous extension of this map according to Proposition 2.40. Let $\phi \colon X \to [0,1]$ be a continuous map taking the value 1 on A and the value 0 on $F^{-1}(\{-1,1\})$. Define $F = \bar{F} \cdot \varphi \colon X \to [-1,1]$ and observe that this map has image in $]-1,1[\cong \mathbb{R}$. Composed with this homeomorphism, the resulting map has the desired properties. □

2.4. **Profinite spaces.**

2.41. **Definition** A profinite space is a cofiltered inverse limit in Top of finite discrete topological spaces.

2.42. **Remark** We will see below, that we could equivalently define profinite spaces as general inverse limits in Top of discrete spaces. This is a consequence of the proof of Theorem 2.48 below.

The following constitutes canonical examples of profinite spaces, indeed of very special profinite spaces:

2.43. **Definition** A profinite group is a topological group, whose underlying topological space is profinite.

2.44. **Definition** The profinite completion $\widehat{\Gamma}$ of a (discrete) group Γ is the group

$$\lim_{\Gamma \to Q} Q$$

where the limit runs over the system of finite quotients (with maps under Γ) or equivalently over the poset of all finite index subgroups of Γ. By construction there is a group homomorphism $\Gamma \to \widehat{\Gamma}$ and $\widehat{\Gamma}$ is a profinite group.

Thanks to Simon Weinzierl for pointing out the following (which I had wrongly claimed in an earlier version of this script).

2.45. **Remark** A profinite group Γ which is is topologically finitely generated satisfies that the map $\Gamma \to \widehat{\Gamma}$ is a continuous bijection, and therefore a homeomorphism since both Γ and $\widehat{\Gamma}$ are compact Hausdorff. This follows from a deep result of Nikolov–Segal [**?**] which shows that finite index subgroups of Γ are open (the proof relies on the classification of finite simple groups!)

In general, finite index subgroups of profinite groups need not be open, so the map $\Gamma \to \widehat{\Gamma}$ is in general not continuous, and in particular not a homeomorphism.

For instance, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is a profinite group which has non-open finite index subgroups, see [**?**, Chapter 7]. In particular, it is not topologically finitely generated.

2.46. **Example** The p-adic integers $\mathbb{Z}_p$, the profinite integers $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. More generally Galois groups of infinite field extensions (like $\mathbb{F}_p \to \overline{\mathbb{F}}_p$ whose Galois group is $\widehat{\mathbb{Z}}$), or even more generally the étale fundamental group of a scheme.

2.47. **Remark** A (discrete) group Γ for which the map $\Gamma \to \widehat{\Gamma}$ is injective is called residually finite. Examples of such contain free groups, finitely generated nilpotent group, finitely generated linear groups, and fundamental groups of compact 3-manifolds.

Among such residually finite groups there is a question (often studied in the context of 3-manifold groups) about profinite rigidity: It asks whether for two residually finite groups Γ_1 and Γ_2 with $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$, is there an isomorphism $\Gamma_1 \cong \Gamma_2$? Spectacular positive results have quite recently been given by Bridson–McReynolds–Reid–Spitler [**?**].

The following theorem provides many more examples of profinite spaces.

2.48. **Theorem** *Let X be a topological space. Then the following are equivalence*

(1) *X is profinite, and*
(2) *X is totally disconnected, compact Hausdorff.*

Proof. We will first show that the totally disconnected spaces, and compact Hausdorff spaces are closed under limits in Top. Indeed, we first note that totally disconnected spaces, compact spaces and Hausdorff spaces are closed under products: So let X_i be a family of topological spaces. If each X_i is totally disconnected and $A \subseteq \prod_i X_i$ is a connected subspace, then $p_i(A) \subseteq X_i$ is also connected by Lemma 2.9. Hence $p_i(A) = \{x_i\}$. Since this holds for all $i \in I$, we find that A is itself a singleton $A = \{(x_i)_{i\in I}\}$. Thus products of totally disconnected spaces are totally disconnected. If all X_i are compact, then Tychonoff's theorem Theorem 2.17 tells us that $\prod_I X_i$ is compact, and we leave as an exercise to show that if all X_i are Hausdorff, then so is the product.

Now, the limit of a general diagram $X\colon I \to \mathrm{Top}$ is a subspace of the product $\prod_I X_i$. Thus it suffices to note that subspaces of totally disconnected spaces are totally disconnected (exercise). Likewise, a subspace of a Hausdorff space is again Hausdorff. Now, in general, a subspace of a compact space need not be compact (of course, but find an example as an exercise), however, we have seen that closed subspaces of compact spaces are compact. It therefore suffices to show that the limit of a diagram $I \to \mathrm{Top}$ where all X_i are compact Hausdorff, is a closed subspace of the product $\prod_I X_i$. For this we recall that the limit was an intersection of subspaces of the form $\Gamma_{i\to j} \times \prod_{k\neq i,j} X_k$, so it suffices to note that $\Gamma_{i\to j}$ is closed. For this, we refer to again to the exercise below. In summary, we have shown (more than) the implication (1) $\Rightarrow$ (2): Finite discrete spaces are totally disconnected, compact Hausdorff and profinite spaces are defined as (certain) inverse limits over finite discrete spaces.

Let us now show the implication (2) $\Rightarrow$ (1). So let X be a topological space. Consider the following partially ordered set $\mathfrak{I}$. The elements of $\mathfrak{I}$ are the finite sets I such that there is a (disjoint) decomposition $X = \amalg_{i\in I} U_i$ into open (and therefore also closed) subspaces U_i. A partial order on $\mathfrak{I}$ is defined by refinement: $I \leq I'$ if for all $i' \in I$ there is an $i \in I$ such that $U_{i'}$ is an open subspace of U_i, i.e. the cover of X associated to I' refines the cover associated to I). This partial order is cofiltered, as one can always find a common refinement of two covers. For any $I \in \mathfrak{I}$ there is a canonical continuous map $X \to I^\delta$ by collapsing each open to a point:

$$X = \coprod_{i\in I} U_i \longrightarrow \coprod_{i\in I} * = I.$$

It is immediate from the definitions that this map is compatible with refinements, so we obtain a canonical continuous map

$$\Phi\colon X \longrightarrow \lim_{I\in\mathfrak{I}} I$$

whose image is profinite by construction. We will now show that when X is totally disconnected compact Hausdorff, this map is a homeomorphism. In fact, by Lemma 2.34 it suffices to show that Φ is bijective. As before, we denote by $C(x)$ the connected component of x in X. We will show that in general, $\Phi(x) = \Phi(x')$ if and only if $x' \in C(x)$. It follows that if X is totally disconnected, the map Φ is injective, since then $C(x) = \{x\}$. So let Φ' be the composite

$$X \xrightarrow{\Phi} \lim_{I \in \mathfrak{J}} I \longrightarrow \prod_{I \in \mathfrak{J}} I.$$

We have $\Phi(x) = \Phi(x')$ if and only if $\Phi'(x) = \Phi'(x')$ since the latter of the two above maps is injective. However, we have $\Phi'(x) = \Phi'(x')$ if and only if for every open and disjoint finite decomposition $X = \amalg_I U_i$ we have that $x \in U_i$ if and only if $x' \in U_i$. In other words, if and only if x' is contained in every open and closed subspace which contains x. By Lemma 2.36 we deduce that $x' \in C(x)$ as needed.

It remains to show that Φ is surjective, i.e. we need to show that for each $t \in \lim_{I \in \mathfrak{J}} I$ we find that $\Phi^{-1}(t)$ is non-empty. For $J \in \mathfrak{J}$, we denote by

$$p_J \colon \lim_{I \in \mathfrak{J}} I \subseteq \prod_{I \in \mathfrak{J}} I \longrightarrow J$$

the canonical projection. We then observe, that

$$\{t\} = \bigcap_{I \in \mathfrak{J}} p_I^{-1} p_I(t).$$

Indeed, the inclusion $\subseteq$ is true by definition. To see the other inclusion, it suffices to see that $t = (t_I)_{I \in \mathfrak{J}}$ where $t_I = p_I(t) \in I$, so that t is determined by its components. Then we have

$$\Phi^{-1}(t) = \bigcap_{I \in \mathfrak{J}} \Phi_I^{-1}(t_I)$$

where $\Phi_I = p_I \circ \Phi$. By construction, $\Phi_I \colon X \to I$ is surjective, so $\Phi_I^{-1}(t_I)$ is not empty. Moreover, for each finite subset $\{I_1, \dots, I_n\} \subseteq \mathfrak{J}$, we find that the intersection

$$\Phi_{I_1}^{-1}(t_1) \cap \cdots \cap \Phi_{I_n}^{-1}(t_n)$$

is also not empty: After suitable refinement, we find $J \in \mathfrak{J}$ with $I_i \leq J$ for all $i = 1, \dots, n$. Then we find $\Phi_J^{-1}(t_J) \subseteq \Phi_{I_i}^{-1}(t_i)$ for all $i = 1, \dots, n$, and as just observed, $\Phi_J^{-1}(t_J)$ is not empty. Appealing to Lemma 2.16 we deduce that $\Phi^{-1}(t)$ is not empty, so that Φ is surjective. □

Exercise. Show the following assertions.

(1) Subspaces of totally disconnected spaces are totally disconnected.
(2) Products of Hausdorff spaces are Hausdorff.
(3) Let $f \colon X \to Y$ with Y Hausdorff be a map of sets between topological spaces. If f is continuous, then $\Gamma_f \subseteq X \times Y$ is closed. If Y is compact Hausdorff and $\Gamma_f \subseteq X \times Y$ is closed, then f is continuous. Show that these assumptions are necessary.

2.49. **Definition** A topological space is called extremally disconnected if for every open set $U \subseteq X$ its closure $\overline{U}$ is again open.

2.50. **Lemma** *Extremally disconnected Hausdorff spaces are totally disconnected.*

Proof. Let X be an extremally disconnected Hausdorff space and $x \in X$. We wish to show that $C(x) = \{x\}$. So let $y \neq x$, we aim to show that $y \notin C(x)$. Choose disjoint opens $U \ni x$ and $V \ni y$. Since $y \notin X \setminus V$, $U \subseteq X \setminus V$ and $X \setminus V$ is closed, we deduce that $y \notin \overline{U}$. Since X is extremally disconnected, $\overline{U}$ is open and closed. Consequently, $\overline{U} \cap C(x)$ is non-empty (it contains x) and an open and closed subspace of $C(x)$. It is therefore equal to $C(x)$. Consequently, $y \notin \overline{U} \cap C(x) = C(x)$, and consequently we deduce that $C(x) = \{x\}$. □

2.51. **Remark** In general, extremally disconnected spaces need not be totally disconnected. Indeed, consider an infinite set X with the *cofinite* topology, where a set is closed if and only if it is finite or all of X. This is always extremally disconnected: Let $U \subseteq X$ be open. Then $U \subseteq \overline{U}$ and therefore $X \setminus \overline{U} \subseteq X \setminus U$ and the latter is finite, hence so is the former and therefore $\overline{U}$ is open. Now we claim that X is connected. So let $A \subseteq X$ be a non-empty open and closed subset. This means that both A and $X \setminus A$ are closed, which can only be true if $A = X$.

2.52. **Lemma** *Let U and V be open and disjoint subspaces of an extremally disconnected space X. Then $\overline{U}$ and $\overline{V}$ are again disjoint.*

Proof. By assumption, $X \setminus Y$ is closed and contains V. Therefore $\overline{V} \subseteq X \setminus U$, in other words $\overline{V} \cap U = \varnothing$. Since X is extremally disconnected, $\overline{V}$ is again open. The same argument then shows that $\overline{V} \cap \overline{U} = \varnothing$. □

Next, we will prove a theorem of Gleason which characterises the extremally disconnected compact Hausdorff spaces.

2.53. **Theorem** *Let X be a compact Hausdorff space. The following are equivalent:*

(1) *X is extremally disconnected, and*
(2) *for every compact Hausdorff space Y and every continuous surjection $f\colon Y \to X$, there exists a continuous section $s\colon X \to Y$.*

Proof. We begin with the implication (2) ⇒ (1). So let $U \subseteq X$ be an open subspace. We need to show that its closure $\overline{U}$ is again open. Consider the (disconnected) space

$$Y = \big[(X \setminus U) \times \{0\}\big] \cup \big[\overline{U} \times \{1\}\big] \subseteq X \times \{0, 1\}$$

together with its map $Y \to X \times \{0, 1\} \to X$ where the latter map is the canonical projection. This map is continuous and surjective, and hence admits a continuous section $s\colon X \to Y$. We find that $s(U) \subseteq X \times \{1\}$, so that by continuity of s, we also have $s(\overline{U}) \subseteq X \times \{1\}$. Then we obtain $\overline{U} = s^{-1}(\overline{U} \times \{1\})$ which is open since s is continuous and $\overline{U} \times \{1\} \subseteq Y$ is open.

Next, we show the implication (1) $\Rightarrow$ (2). So let X be extremally disconnected and $f\colon Y \to X$ a continuous surjection. Consider the set

$$\{A \subseteq Y \quad | \quad A \text{ closed and } f(A) = X\}.$$

By Zorn's lemma, this set contains a maximal element, which we again call A. Thus we have $A \subseteq Y$ closed, $f(A) = X$ and $f(A') \neq X$ for any strict inclusion $A' \subseteq A$ with A' closed. We claim that $f_{|A}\colon A \to X$ is a homeomorphism. A section of f is then given by the composite

$$X \xrightarrow{(f_{|A})^{-1}} A \longrightarrow Y.$$

Now since A is a closed subset of the compact space Y we know that A is compact. Since X is Hausdorff, Lemma 2.34 implies that it suffices to show that $f_{|A}$ is injective (as it is surjective by definition of A). So let $a \neq a'$ two distinct points of A. We aim to show that $f(a) \neq f(a')$. Since A is Hausdorff, we can choose disjoint opens $U, U' \subseteq A$ with $a \in U$ and $a' \in U'$. Then

$$X \setminus f(A \setminus U) \quad \text{and} \quad X \setminus f(A \setminus U')$$

are open, as follows again from Lemma 2.34 which gives that f is closed. In addition, we have

$$\big[X \setminus f(A \setminus U)\big] \cap \big[X \setminus f(A \setminus U')\big] = \varnothing$$

since $U \cap U' = \varnothing$. From Lemma 2.52, we infer that

$$\big[\overline{X \setminus f(A \setminus U)}\big] \cap \big[\overline{X \setminus f(A \setminus U')}\big] = \varnothing.$$

We will now show that

$$f(a) \in \overline{X \setminus f(A \setminus U)} \text{ and } f(a') \in \overline{X \setminus f(A \setminus U')}$$

which together with the above implies that $f(a) \neq f(a')$.

It now suffices to show that for each open set $V \subseteq X$ with $f(a) \in V$, we have $V \cap \big[X \setminus f(A \setminus U)\big] \neq \varnothing$. Indeed, if $f(a) \notin \overline{X \setminus f(A \setminus U)}$, then $f(a) \in X \setminus \overline{X \setminus f(A \setminus U)}$, which is an open set whose intersection with $X \setminus f(A \setminus U)$ is empty.

We now consider the open subspace $U \cap f^{-1}(V)$ of A. This is non-empty since it contains a. Therefore, $A' = A \setminus (U \cap f^{-1}(V))$ is a proper and closed subspace of A. By the maximality of A, we know that the restriction of f to A' is not surjective. So let

$$x \in X \setminus f(A') = X \setminus f(A \setminus U \cap f^{-1}(V)) \subseteq X \setminus f(A \setminus U).$$

Since $f_{|A}$ is surjective, there is $b \in A$ with $f(b) = x$. By construction, $b \in U \cap f^{-1}(V)$. Then we also find that $x = f(b) \in f(f^{-1}(V)) = V$. Together we find that

$$x \in V \cap \big[X \setminus f(A \setminus U)\big]$$

so the required intersection is non-emtpy. $\square$

Exercise. Prove or disprove the following statements:

1. A subspace of an extremally disconnected space is extremally disconnected.
2. An open subspace of an extremally disconnected space is extremally disconnected.
3. A closed subspace of an extremally disconnected space is extremally disconnected.
4. A dense subspace of an extremally disconnected space is extremally disconnected.

Exercise. Show that extremally disconnected compact Hausdorff spaces are *projective* in the category CH of compact Hausdorff spaces, that is for each solid diagram, the dashed arrow as indicated in the below diagram exists, rendering the diagram commutative:

$$\begin{tikzcd} & Z \arrow[d, two heads, "f"] \\ X \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Here, X is extremally disconnected compact Hausdorff, Y and Z are compact Hausdorff, and f is surjective.

We add here for completeness the following result due to Rudin.

2.54. **Proposition** *Let X and Y be infinite extremally disconnected compact Hausdorff spaces. Then $X \times Y$ is not extremally disconnected.*

Proof. Since X and Y are compact Hausdorff, one can find continuous functions $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$ with $f(X)$ and $g(Y)$ infinite subsets of $\mathbb{R}$. The following argument might be overkill, but does the job: A continuous function $f\colon X \to \mathbb{R}$ is precisely a self-adjoint element of the C^*-algebra $C(X)$. It is an observation of Kaplansky's that any infinite dimensional C^*-algebra contains a self-adjoint element with infinite spectrum. Now, the spectrum of the element $f \in C(X)$ turns out to be $f(X)$, so the claim follows from the exercise that $C(X)$ is finite dimensional if and only if X is finite. Hint: you can use Tietze's extension theorem.

Another argument is as follows: The statement we seek to show can be shown to be true for metrisable compact Hausdorff spaces (exercise). It therefore suffices to find a surjection from X onto an infinite metrisable compact Hausdorff space. For this, (for instance) choose a sequence $x_1, x_2, \dots$ of pairwise distinct elements in X. Separate x_i from x_{i+1} by a function $f_i\colon X \to [0,1]$ (this can be done by Urysohn's lemma). These functions provide a map $X \to \prod_{i=1,2,\dots} [0,1]$. The image of this space is infinite (since the resulting function distinguishes each of the points x_i) and the space $[0,1]^{\mathbb{N}}$ is metrisable. The image of the map $X \to [0,1]^{\mathbb{N}}$ is closed and hence a compact Hausdorff space, and metrisable as a subspace of a metrisable space.

Let $\{J_i\}_{i=1,2,\dots}$ be a sequence of open disjoint intervals with $J_i \cap f(J_i) \neq \varnothing$. Set $A_i = f^{-1}(J_i)$. Define $B_i \subseteq Y$ similarly. Set

$$V = \bigcup_{i=1,2,\dots} A_i \times B_i \quad \text{and } W = \bigcup_{j \neq k} A_j \times B_k$$

which are open and disjoint subsets of $X \times Y$. By Lemma 2.52, it suffices to show that $\overline{V} \cap \overline{W} \neq \varnothing$. Assume to the contrary that $\overline{V} \cap \overline{W} = \varnothing$. Consider an open cover $\{U_i\}_{i \in I}$ of $\overline{V}$. Intersecting with the open set $X \setminus \overline{W}$, we may assume that the open cover intersects W trivially. After refining U_i by basic open sets and using the compactness of $\overline{V}$, we may find finitely many opens $C_1, \dots, C_n \subseteq X$ and $D_1, \dots, D_n \subseteq Y$ such that

$$V \subseteq \bigcup_{i=1,\dots,n} C_i \times D_i.$$

By construction, it must be that there is an $i \in \{1, \dots, n\}$ such that $C_i \times D_i$ intersects $A_j \times B_j$ and $A_k \times B_k$ for $j \neq k$ in a non-trivial way. But then $C_i \times D_i$ intersects $A_j \times B_k$ in a non-trivial way and therefore $C_i \times D_i \cap W \neq \varnothing$, a contradiction. $\square$

2.55. **Remark** Among non compact spaces, the situation is of course different: We have that for two sets M and N, that

$$M^\delta \times N^\delta \cong (M \times N)^\delta.$$

Therefore, the product of two general extremally disconnected spaces can easily be extremally disconnected.

Also, a finite compact Hausdorff space X is discrete, and therefore $X \times Y$ is homeomorphic to a finite disjoint union of copies of Y. Consequently, if Y is extremally disconnected compact Hausdorff and X is finite, $X \times Y$ is extremally disconnected compact Hausdorff.

The next goal is to find a sufficient supply of extremally disconnected spaces. For this, we will use the Stone-Cech compactification.

2.56. **Definition** Let X be a topological space. A Stone-Cech compactification of X is a compact Hausdorff space $\beta(X)$ together with a continuous map $\iota_X \colon X \to \beta(X)$ satisfying the following universal property:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota_X"'] & Y \\ \beta(X) \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

For every continuous map $f \colon X \to Y$, with Y a compact Hausdorff space, there is a unique extension of f along ι_X to $\beta(X)$.

2.57. **Theorem** *Every topological space admits a Stone-Cech compactification.*

We will give one construction of Stone-Cech compactifications for discrete spaces. The general existence result was discussed in the exercises (ask Philipp for notes).

Proof. Let M be a set and let $\beta(M)$ denote the set of ultrafilters on M. For a subset $A \subseteq M$, we consider the subset of $\beta(M)$ given by

$$[A] = \{\mathcal{F} \in \beta(M) \mid A \in \mathcal{F}\}.$$

We have the following basic properties of the association $A \mapsto [A]$ (do this as exercise):

(1) $[\varnothing] = \varnothing$ and $[M] = \beta(M)$.
(2) $[A] \subseteq [B] \Leftrightarrow A \subseteq B$,
(3) $[A] = [B] \Leftrightarrow A = B$,
(4) $[A] \cup [B] = [A \cup B]$
(5) $[A] \cap [B] = [A \cap B]$
(6) $[M \setminus A] = \beta(M) \setminus [A]$.

One can deduce from the above properties that the collection $\{[A]\}_{A \subseteq M}$ of subsets of $\beta(M)$ is the basis of a topology on $\beta(M)$. We call the resulting topological space $\beta(M)$, though more precisely, we should perhaps write $\beta(M^\delta)$. We note that the above properties imply that the sets $[A]$ are open and closed in $\beta(M)$. We first show that $\beta(M^\delta)$ is a Hausdorff space. So let $\mathcal{F} \neq \mathcal{G}$ be ultrafilters on M and let $A \in \mathcal{F}$ such that $A \notin \mathcal{G}$ (such an A exists, since $\mathcal{F} \neq \mathcal{G}$ and both are ultrafilters so neither can be contained in the other). Since $\mathcal{G}$ is an ultrafilter, we find that $M \setminus A \in \mathcal{G}$ by Lemma 2.21, so $[A]$ and $[M \setminus A]$ are separating open sets (use property (6) above). Next we show that $\beta(M^\delta)$ is compact. So let consider an open cover, without loss of generality of the form

$$\beta(M^\delta) = \bigcup_{i\in I}[A_i]$$

for some subsets $A_i \subseteq M$. Suppose that there is no finite subcover, i.e. that for all finite $J \subseteq I$, we have

$$\beta(M^\delta) \neq \bigcup_{j\in J}[A_j] = [\bigcup_{j\in J} A_j].$$

We conclude that for all such finite J, we have

$$\bigcap_{j\in J} M \setminus A_j \neq \varnothing$$

and hence by Lemma 2.19 the set $\{M \setminus A_i\}_{i\in I}$ of subsets of M generates a filter, and thus is also contained in an ultrafilter $\mathcal{F}$. Let $i \in I$ be such that $\mathcal{F} \in [A_i]$. Then $A_i \in \mathcal{F}$ but also $M \setminus A_i \in \mathcal{F}$, a contradiction.

It remains to show that $\beta(M^\delta)$ satisfies the claimed universal property. So let $f\colon M \to Y$ be a map from M to a compact Hausdorff space Y. We need to show that there exists a unique continuous map $\bar{f}\colon \beta(M) \to Y$ such that $\bar{f} \circ \iota_M = f$. We will first treat the uniqueness part for which we aim to show that $\overline{M} = \beta(M)$, i.e. that M is dense in $\beta(M)$. By definition, this means that every closed subspace of $\beta(M)$ which contains M must be $\beta(M)$. Equivalently, it means that every open subspace of $\beta(M)$ which is disjoint from M is empty. Hence, it suffices to show that every non-empty open set intersects M in a non-empty manner. It suffices to show this for a basis of the topology. So let $A \subseteq M$ be a non-empty subset. Let $a \in A$. Then $\mathcal{U}(a) = \iota_X(a) \in [A]$. Therefore, $[A] \cap M \neq \varnothing$ as claimed. By the exercise below, there is then at most one continuous extension of f to $\beta(M)$. It finally remains to show the existence of an extension $\bar{f}$ of f.

For this, let $\mathcal{F} \in \mathcal{U}(M) = \beta(M^\delta)$. Then $f_*(\mathcal{F})$ is an ultrafilter on Y, and hence converges to a unique limit point $\bar{f}(\mathcal{F}) \in Y$. When $\mathcal{F} = \mathcal{U}(m)$, we have that $f_*(\mathcal{U}(m)) = \mathcal{U}(f(m))$: By definition $V \subseteq Y$ is contained in $f_*(\mathcal{U}(m))$ if and only if $f^{-1}(V) \in \mathcal{U}(m)$, i.e. when $f^{-1}(V)$ contains m, or in other words when $f(m) \in V$. We therefore find that $\bar{f}(\mathcal{U}(m)) = f(m)$, so $\bar{f}$ is indeed an extension of f. It remains to prove that $\bar{f}$ is continuous.

For this, let $V \subseteq Y$ be an open set. It suffices to show that for each $\mathcal{F}$ with $\bar{f}(\mathcal{F}) \in V$, there is an open subspace $U(\mathcal{F}) \subseteq \bar{f}^{-1}(V)$ which contains $\mathcal{F}$ since then we have

$$\bar{f}^{-1}(V) = \bigcup_{\mathcal{F}\in\bar{f}^{-1}(V)} U(\mathcal{F}).$$

So let $\mathcal{F}$ be such that $\bar{f}(\mathcal{F}) \in V$. By an exercise below, we can find an open subset $W \subseteq V$ with $\overline{W} \subseteq V$ and $\bar{f}(\mathcal{F}) \in W$. By definition, $\bar{f}(\mathcal{F})$ is the limit point of $f_*(\mathcal{F})$. Since W is open and contains $\bar{f}(\mathcal{F})$, we conclude that $W \in f_*(\mathcal{F})$, i.e. that $f^{-1}(W) \in \mathcal{F}$. Therefore, $\mathcal{F} \in [f^{-1}(W)]$ and $[f^{-1}(W)]$ is open.

Moreover, we have

$$\bar{f}([f^{-1}(W)]) = \bar{f}(\overline{f^{-1}(W)}) \subseteq \overline{f(f^{-1}(W))} = \overline{W} \text{ and therefore } [f^{-1}(W)] \subseteq \bar{f}^{-1}(\overline{W}) \subseteq \bar{f}^{-1}(V).$$

again by an exercise below. $\square$

Exercise. Let $A \subseteq M$ be a subset. Then show that $\overline{A} \subseteq \beta(M)$ coincides with $[A]$.

Exercise. Given what we have shown, prove that there is at most one continuous extension of $f\colon M \to Y$ to $\beta(M)$.

Exercise. Let X be a compact Hausdorff space. Consider an open set U which contains x. Show that there is an open set V also containing x and such that $\overline{V} \subseteq U$.

Exercise. Let X be a set and $\mathcal{P}(X)$ its power set, i.e. the set of subsets of X.

(1) Show that there is a bijection $\mathcal{P}(X) \to \{0,1\}^X$, given by sending $Y \subseteq X$ to the function
$$x \mapsto \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}.$$
(2) Show that the set of ultrafilters on X is a subset of $\mathcal{P}(\mathcal{P}(X))$, and therefore of $\{0,1\}^{\mathcal{P}(X)}$. Deduce that $|\beta(M)| < 2^{2^{|M|}}$.
(3) Equip $\{0,1\}^{\mathcal{P}(X)}$ with the product topology. Show that $\beta(X^\delta)$ has the subspace topology of the product topology. Show that $\beta(X^\delta)$ is a closed subset. Deduce that $\beta(X^\delta)$ is totally disconnected, compact Hausdorff.

2.58. **Theorem** *The Stone-Cech compactification β assembles into a left adjoint of the inclusion* $\mathrm{CH} \subseteq \mathrm{Top}$.

2.59. **Remark** One can show that the adjunction $\mathrm{CH} \leftrightarrows \mathrm{Top}$ is monadic: The forgetful functor is fully faithful (and hence conservative) and preserves split coequlisers (this is a variant of the fact that for X compact Hausdorff and $A \subseteq X$ a closed subspace, X/A is again compact Hausdorff), and it admits a left adjoint Theorem 2.58. One may then apply Beck's monadicity theorem. The monad associated to this adjunction is sometimes called the ultrafilter monad.

Exercise. Prove or disprove that the inclusion $\mathrm{CH} \to \mathrm{Top}$ admits a right adjoint.

2.60. **Lemma** *Let M be a set. Then $\beta(M^\delta)$ is extremally disconnected.*

Proof. By Theorem 2.53 it suffices to show that any continuous surjection $f\colon Y \to \beta(M)$ where Y is compact Hausdorff, admits a continuous section $s\colon \beta(M) \to Y$. For $m \in M \subseteq \beta(M)$ let $y_m \in Y$ be such that $f(y_m) = m$ (this can be done since f is surjective). The association $m \mapsto y_m$ assembles into a (continuous) map $s'\colon M^\delta \to Y$. By the universal property of $\beta(M^\delta)$, it extends uniquely to a map $s\colon \beta(M) \to Y$. It remains now to show that the resulting composite
$$\beta(M) \xrightarrow{s} Y \xrightarrow{f} \beta(M)$$
is the identity. Again, it follows from the universal property that it suffices to show that the composite
$$M \xrightarrow{\iota_M} \beta(M) \xrightarrow{s} Y \xrightarrow{f} \beta(M)$$
is the canonical map ι_M again. This is true by construction of the map s: restricted to the image of ι_M it is a section of f. □

2.61. **Corollary** *Every topological space admits a surjection from an extremally disconnected space. In particular, every compact Hausdorff space is a quotient of an extremally disconnected compact Hausdorff space.*

Proof. Consider the map $\beta(X^\delta) \to X$ induces by the continuous (and surjective) map $X^\delta \to X$ and use Lemma 2.60. The final claim follows from Corollary 2.35. $\square$

Exercise. Show that every extremally disconnected compact Hausdorff space is a retract of the Stone-Cech compactification of a discrete set.

3. Sites and sheaves

In the following we wish to talk about sheaves on categories (which we think of as generalisations of the category of open sets on a topological space). In order to do so, we will introduce various notions of coverings leading to the definition of a Grothendieck topology on a small category.

First, we recall the simplex category - it is a basic and important indexing category. For instance, it is essential in (more than one) of the foundations of ∞-category theory and plays a traditional role in homotopy theory and algebraic geometry (the latter for its appearance concerning sheaves as we will see below).

3.1. **Definition** The category Δ is the full subcategory of the category of partially ordered sets on the linearly ordered sets $[n] = \{0, \ldots, n\}$. In other words, maps in Δ from $[n]$ to $[m]$ are the (weakly) monotone maps. We let Δ_{inj} be the wide (i.e. containing all objects) subcategory on the strictly monotone (i.e. in addition injective) maps. For $n \geq 0$, we let $\Delta_{(\mathrm{inj}),\leq n}$ be the full subcategory of $\Delta_{(\mathrm{inj})}$ on objects $[k]$ with $k \leq n$. Functors $X \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ are called simplicial objects in $\mathcal{C}$, functors $Y \colon \Delta \to \mathcal{C}$ are called cosimplicial objects. Likewise, functors $X \colon \Delta_{\mathrm{inj}}^{(\mathrm{op})} \to \mathcal{C}$ are called semi-(co)-simplicial objects.

We will be concerned with many (co)-simplicial diagrams and their (co)limits. It turns out that (co)limits of (co)simplicial objects in ordinary categories (rather than ∞-categories) can be described in much more elementary terms. The following is the general result:

Thanks to Maxime Ramzi and Denis Nardin for suggestions for the following arguments.

3.2. **Lemma** *The functor $\Delta_{\mathrm{inj},\leq k} \to \Delta$ is k-coinitial. That is, for a simplicial object $X \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ with $\mathcal{C}$ a k-category*[1] *the canonical map*

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}_{\mathrm{inj},\leq k}} X \longrightarrow \operatorname*{colim}_{\Delta^{\mathrm{op}}} X$$

is an equivalence. Likewise, for any cosimplicial object $Y \colon \Delta \to \mathcal{C}$, the canonical map

$$\lim_{\Delta} Y \longrightarrow \lim_{\Delta_{\mathrm{inj},\leq k}} Y$$

is an equivalence.

3.3. **Lemma** *Let $i \colon K \to L$ be a functor between small ∞-categories and assume that for all objects $x \in L$, the slice $K_{x/}$ is k-connective. Then, for every k-category $\mathcal{E}$, and every diagram $F \colon L \to \mathcal{E}$, the canonical map*

$$\operatorname*{colim}_{K} Fi \to \operatorname*{colim}_{L} F$$

is an equivalence. Dually, if the slices $K_{/x}$ are contractible, then the resulting map

$$\lim_{L} F \to \lim_{K} Fi$$

is an equivalence.

[1] Here, a k-category is an ∞-category all whose mapping anima are $(k-1)$-truncated. In particular, a 1-category is an ordinary category as we are used to.

Proof. We first show the statements of the lemma in case $\mathcal{E}$ is bicomplete. We wish to show that the following diagram commutes.

$$\begin{tikzcd} \mathrm{Fun}(L,\mathcal{E}) \arrow[rr, "i^*"] \arrow[dr, "\mathrm{colim}"'] & & \mathrm{Fun}(K,\mathcal{E}) \arrow[dl, "\mathrm{colim}"] \\ & \mathcal{E} & \end{tikzcd}$$

Equivalently, we may pass to the diagram of right adjoints and show that this commutes (this is where we use that $\mathcal{E}$ is complete). This is given by

$$\begin{tikzcd} \mathrm{Fun}(L,\mathcal{E}) & & \mathrm{Fun}(K,\mathcal{E}) \arrow[ll, "i_*"'] \\ & \mathcal{E} \arrow[ul, "\mathrm{diag}"] \arrow[ur, "\mathrm{diag}"'] & \end{tikzcd}$$

The diagonal functor takes an object $e \in \mathcal{E}$ to the functor $K \to * \xrightarrow{e} \mathcal{E}$, and similarly for L. By the formula for right Kan extensions (exercise below) we have

$$i_*(\mathrm{diag}(e))(d) = \lim_{K_{d/}} \mathrm{diag}(e).$$

Now, we claim that the latter limit is given, via the canonical map from e, by e. Indeed, we calculate that for any other object $e' \in \mathcal{E}$, we have

$$\mathrm{Map}_{\mathcal{E}}(e', \lim_{K_{d/}} e') = \lim_{K_{d/}} \mathrm{Map}(e', e)$$

so it suffices to show that the functor $\mathcal{I} \to \mathrm{An}_{\leq k-1}$ which is constant at an object X is given by $\mathrm{Map}(|\mathcal{I}|, X)$. This follows immediately from the straigthening-unstraightening equivalence, which describes the limit as the cocartesian sections of the cocartesian fibration classified by the functor $\mathcal{I} \to \mathrm{An}$. Now, since this functor is constant, the cocartesian fibration is given by $\mathcal{I} \times X \to \mathcal{I}$. Therefore, cocartesian sections are given by the category of functors $\mathcal{I} \to X$. Since X is an ∞-groupoid, we obtain the described formula for the limit. Then we note that $\mathrm{Map}(Y, X)$ is contractible if X is $(k-1)$-truncated and Y is $(k-1)$-connected, i.e. k-connective.

The dual statement allows for the same proof: We consider the diagram with i^* and the limit functors. Now, we show that the diagram of left adjoints commutes. This then amounts to showing that $i_!(\mathrm{diag}(e))$ is again a constant functor with value e. The argument for this is similar as before, using that in anima we have the equivalence $\tau_{\leq k-1}(X \times Y) \simeq \tau_{\leq k-1}X \times \tau_{\leq k-1}Y$ and that $\tau_{\leq k-1}X$ vanishes if X is k-connective.

The argument just given shows the conclusion of the lemma in particular for $\mathcal{E} = \mathrm{An}_{\leq k-1}$. Now in general, let $F\colon \mathcal{D} \to \mathcal{E}$ be a functor. To prove the general version of the above lemma, by the Yoneda lemma, it suffices to show that for all objects $e \in \mathcal{E}$, the canonical map

$$\mathrm{Map}_{\mathcal{E}}(\operatorname*{colim}_{\mathcal{D}} F, e) \to \mathrm{Map}_{\mathcal{E}}(\operatorname*{colim}_{\mathcal{C}} Fi, e)$$

is an equivalence. But this map is equivalent to the map

$$\lim_{\mathcal{D}^{\mathrm{op}}} \mathrm{Map}_{\mathcal{E}}(F(-), e) \to \lim_{\mathcal{C}^{\mathrm{op}}} \mathrm{Map}_{\mathcal{E}}(Fi(-), e).$$

Now, consider the composite

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{i} \mathcal{D}^{\mathrm{op}} \xrightarrow{F} \mathcal{E}^{\mathrm{op}} \xrightarrow{\mathrm{Map}_{\mathcal{E}}(-,e)} \mathrm{An}_{\leq k-1}$$

where the last functor is well-defined since $\mathcal{E}$ is a k-category. We have that $(\mathcal{C}^{\mathrm{op}})_{/d} \simeq (\mathcal{C}_{d/})^{\mathrm{op}}$. Therefore, the resulting map on limits is an equivalence by the first step. The argument for the maps between limits is similar. □

To prove Lemma 3.2, it now suffices to show the following lemma.

3.4. **Lemma** *Let $k \geq 0$. Then the functors $\Delta_{(\mathrm{inj}),\leq k} \to \Delta_{(\mathrm{inj})}$ is k-coinitial.*

Proof. By the previous, it suffices to show that for each $[n] \in \Delta_{(\mathrm{inj})}$, the slice category $(\Delta_{(\mathrm{inj}),\leq k})_{/[n]}$ is k-connective. Let $\mathcal{P}([n];k)$ denote the set of non-empty subsets of $[n]$ of cardinality less or equal to k. This is partially ordered by inclusion. There is a canonical functor

$$\pi\colon (\Delta_{(\mathrm{inj}),\leq k})_{/[n]} \to \mathcal{P}([n];k)$$

sending a map $f\colon [m] \to [n]$ to the Image $f([m])$ of f. This has the correct cardinality bound because $[m] \in \Delta_{(\mathrm{inj}),\leq k}$. We claim that this functor is a cartesian fibration (in both cases, i.e. with inj at both categories and without inj at both categories):

$$\begin{tikzcd} [m'] \arrow[r] \arrow[d] & [m] \arrow[d] \\ [s'] \arrow[r] & [s] \end{tikzcd}$$

and since $[s'] \to [s]$ is injective, so is $[m'] \to [m]$. Moreover, since $[m] \to [s]$ is surjective, so is $[m'] \to [s']$. It is then direct to check that the resulting morphism $[m'] \to [m]$ is π-cartesian. Therefore, the opposite of the cartesian fibration π is cocartesian. In particular, we deduce that the homotopy type of the category $(\Delta_{\mathrm{inj},\leq k})_{/[n]}$ is given by the colimit

$$\operatorname*{colim}_{s \in \mathcal{P}([n];)^{\mathrm{op}}} |\pi^{-1}(s)|.$$

Now, the fibres of π over $s \subseteq [n]$ is given by all objects $([m], f\colon [m] \to [n])$ where $f([m]) = s$, and all morphisms between such objects. In the case with inj at the categories, this fibre consists of a single element (because there is a unique bijective map $[m] \to [s]$). In particular, the functor $\pi\colon (\Delta_{\mathrm{inj},\leq k})_{/[n]} \to \mathcal{P}([n],k)$ is in fact an equivalence of categories. In the case where inj is not part of the categories, the fibre has a terminal object: the identity of $[s]$. In both cases, the fibre categories are contractible. It therefore suffices to show that $|\mathcal{P}([n];k)|$ is k-connective. In fact, we claim that this homotopy type is a wedge of k-spheres. □

To end this detour, we note the better known cofinality result, see [**?**, 6.5.3.7]:

3.5. **Lemma** *The functor $\Delta_{\mathrm{inj}} \to \Delta$ is ∞-coinitial.*

Proof. Let $\mathcal{C} = (\Delta_{\mathrm{inj}})_{/[n]}$. We show that $\mathcal{C}$ is contractible. Consider the functor $F\colon \mathcal{C} \to \mathcal{C}$ sending a map $[m] \xrightarrow{f} [n]$ to the map $[0] \star [m] \xrightarrow{\bar{f}} [n]$ with $\bar{f}(0) = 0$. There is a natural transformation from the identity to F and from the functor which is constant at the object $[0] \to [n]$ (with image 0). Therefore, the identity is homotopic to a constant map which means that $\mathcal{C}$ is contractible. □

Proof of Lemma 3.2. The functor under investigation is the composite

$$\Delta_{\mathrm{inj},\leq k} \to \Delta_{\mathrm{inj}} \to \Delta.$$

The first is k-coinital by Lemma 3.4 and the latter is ∞-coinitial by Lemma 3.5. Consequently the composite is k-coinitial as needed. □

3.6. **Remark** We have shown that also the functor $\Delta_{\leq k} \to \Delta$ is k-coinitial. As a result, the functor $\Delta_{\mathrm{inj},\leq k} \to \Delta_{\leq k}$ is k-coinitial - and as one might expect it is not ∞-coinitial.

Exercise. Prove Lemma 3.2 in the case $k = 1$ directly, i.e. show that the displayed maps are isomorphisms.

Exercise. Calculate $\lim_{\Delta^{\mathrm{op}}} X$ and $\operatorname{colim}_{\Delta} Y$.

3.7. **Definition** Let $\mathcal{C}$ be a small category. A sieve S on an object X is a subfunctor $S \subseteq y(X) = \mathrm{Hom}_{\mathcal{C}}(-, X)$. For $f\colon Y \to X$, we define $f^*(S) = S \times_{y(X)} y(Y)$ where the pullback is performed in $\mathrm{PSh}(\mathcal{C})$. We denote by $\mathrm{Sieve}(X)$ the collection of all sieves on X.

3.8. **Remark** Concretely, for $f\colon Y \to X$ and $S \in \mathrm{Sieve}(X)$, we have that

$$f^*(S)(Z) = \{\alpha\colon Z \to Y \mid f\alpha \in S(Z)\}.$$

3.9. **Definition** Let $\mathcal{C}$ be an essentially small category. A (Grothendieck) topology τ on $\mathcal{C}$ consists of a collection of *covering sieves* $\mathrm{Cov}_\tau(X) \subseteq \mathrm{Sieve}(X)$ for each object X in $\mathcal{C}$ satisfying the following axioms:

(1) For every $X \in \mathcal{C}$ we have $y(X) = \mathrm{Hom}_{\mathcal{C}}(-, X) \in \mathrm{Cov}_\tau(X)$.
(2) For $f\colon Y \to X$ and $S \in \mathrm{Cov}_\tau(X)$ we have $f^*(S) \in \mathrm{Cov}_\tau(Y)$.
(3) Let $S \in \mathrm{Cov}_\tau(X)$ and $R \in \mathrm{Sieve}(X)$. Suppose that for each object $Y \in \mathcal{C}$ and each element $f \in S(Y)$ we have that $f^*(R) \in \mathrm{Cov}_\tau(Y)$. Then $R \in \mathrm{Cov}_\tau(X)$.

A category $\mathcal{C}$ equipped with a topology τ is called a site.

3.10. **Definition** A presheaf F is said to satisfy descent with respect to a sieve S on X if the canonical map $F(X) \to F(S) \overset{\mathrm{def}}{=} \mathrm{Hom}(S, F)$ is an isomorphism. Conversely, we also say that S is F-descendable. A presheaf F on a site $(\mathcal{C}, \tau)$ is called a τ-sheaf, or simply a sheaf if τ is clear from context, if it satisfies descent with respect to all covering sieves of τ.

3.11. **Definition** Let $\mathcal{C}$ be an essentially small category. A quasi-topology ρ consists, for each $X \in \mathcal{C}$, of a collection $\mathrm{Cov}_\rho(X) \subseteq \mathrm{Sieve}(X)$ of sieves on X such that for each $S \in \mathrm{Cov}_\rho(X)$ and each map $f\colon X' \to X$, there exists a sieve $S' \subseteq f^*(S)$ with $S' \in \mathrm{Cov}_\rho(X')$. A category $\mathcal{C}$ equipped with a quasi-topology is called a quasi-site.

3.12. **Definition** A presheaf F on a quasi-site $(\mathcal{C}, \rho)$ is called a ρ-sheaf if it satisfies descent with respect to all sieves contained in ρ.

3.13. **Lemma** *A quasi-topology ρ on a site generates a topology τ, also denoted $\langle \rho \rangle$. The pair $(\mathcal{C}, \tau)$ is called the site associated to the quasi-site $(\mathcal{C}, \rho)$.*

Proof. We say that a topology τ contains the quasi-topology ρ if for all objects X, we have $\mathrm{Cov}_\rho(X) \subseteq \mathrm{Cov}_\tau(X)$ and write $\rho \subseteq \tau$ if τ contains ρ. We then observe that

$$\mathrm{Cov}_{\langle \rho \rangle}(X) = \bigcap_{\tau \text{ s.t. } \rho \subseteq \tau} \mathrm{Cov}_\tau(X)$$

defines covering sieves for a topology $\langle \rho \rangle$ on $\mathcal{C}$ (this follows immediately from the axioms). This is clearly the smallest topology containing ρ. $\square$

We aim to show the following theorem.

3.14. **Theorem** *Let $(\mathcal{C}, \rho)$ be a quasi-site and $(\mathcal{C}, \tau)$ its associated site. Then a presheaf F on $\mathcal{C}$ is a ρ-sheaf if and only if $\mathcal{F}$ is a τ-sheaf.*

The following lemma will be instrumental for this.

3.15. **Lemma** *Let $\mathcal{C}$ be an essentially small category, X an object and F a presheaf on $\mathcal{C}$. Suppose $S \subseteq R$ are sieves on X such that S is F-descendable, and that for all objects Y and all $f \in R(Y)$, there is an F-descendable subsieve $S' \subseteq f^*(S)$. Then R is F-descendable.*

Proof. Let us denote the inclusions by $i \colon S \to R$, $j \colon R \to y(X)$ and write $h = ji$. We consider the maps

$$F(X) \xrightarrow{j^*} F(R) \xrightarrow{i^*} F(S)$$

and wish to show that the map j^* is bijective. Since the composite h^* is bijective by assumption, the map j^* is injective. To show surjectivity, we proceed as follows. We use the exercise below, which implies that

$$F(R) = \lim_{Y \to R} F(Y).$$

That is, elements of $F(R)$ are compatible collections of elements $x_f \in F(Y)$ indexed over the maps $f \colon Y \to R$ (that is, maps $Y \to X$ contained in $R(Y)$). Now consider a compatible collection $\{x_f\}_f$ as just described. By assumption, we have that $i^*(\{x_f\})$ comes from an element $x \in F(X)$, i.e. $h^*(x) = i^*(\{x_f\}_f)$. We now wish to show that $j^*(x) = \{x_f\}_f \in F(R)$. To do so, it suffices to show that for each fixed $f \colon Y \to R$, we have $x_f = f^*(x) \in F(Y)$. To show this, we then consider the diagram

$$\begin{tikzcd}
F(X) \arrow[r, "j^*"] \arrow[rr, bend left, "\cong"] \arrow[dr, "f^*"'] & F(R) \arrow[r, "i^*"] \arrow[d] & F(S) \arrow[d] \\
& F(Y) \arrow[r, "\cong"] & F(S')
\end{tikzcd}$$

where, by construction, the left vertical map takes the collection $\{x_f\}$ to the element $x_f \in F(Y)$. The square in the diagram commutes since it is obtained from a commutative square by applying $\mathrm{Hom}(-, F)$.

By assumption F satisfies descent with respect to S' so the lower horizontal map is also an isomorphism. It therefore suffices to show that $x_f = f^*(x)$ after mapping to $\mathrm{Hom}(S', F) = F(S')$ where it becomes true because the square commutes and x and $\{x_f\}$ have the same image in $\mathrm{Hom}(S, F) = F(S)$ by construction. □

3.16. **Corollary** *Let $(\mathcal{C}, \rho)$ be a quasi-site. Let $S \in \mathrm{Cov}_\rho(X)$ and $R \in \mathrm{Sieve}(X)$ with $S \subseteq R$. Then a ρ-sheaf satisfies descent with respect to R.*

Proof. By Lemma 3.15, it suffices to show that for all objects $Z \in \mathcal{C}$ and $f \in R(Z)$, the sieve $f^*(S)$ contains a sieve $S' \in \mathrm{Cov}_\rho(Z)$. This follows from the definition of a quasi-topology since $S \in \mathrm{Cov}_\rho(X)$ and the fact that $R(Z) \subseteq \mathrm{Hom}_{\mathcal{C}}(Z, X)$. □

Exercise. Let $\mathcal{C}$ be a small category and let F be a presheaf on $\mathcal{C}$. Show that the canonical map

$$\operatorname*{colim}_{X \to F} y(X) \to F$$

is an isomorphism.

The content of the following exercise is often phrased as "colimits in $\mathrm{PSh}(\mathcal{C})$ are universal".

Exercise. Let $\mathcal{C}$ be a small category and let $\mathrm{PSh}(\mathcal{C})$ be its category of presheaves. Let $F \to G$ be a morphism of presheaves and suppose given an isomorphism $\operatorname*{colim}_{i \in I} G_i \cong G$. Set $F_i = F \times_G G_i$. Then show that the canonical map $\operatorname*{colim}_{i \in I} F_i \to F$ is an isomorphism.

While we are at it, we will also need the following result.

Exercise. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between (essentially) small categories. Let $\mathcal{E}$ be a (co)complete category. Then the functor $f^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ has a left/right adjoint denoted by $f_!$ and f^*, respectively. These are the left and right Kan extensions along f. For $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{E})$, they are given by the following formulas

$$f_!(F)(d) = \operatorname*{colim}_{c \in \mathcal{C}_{/d}} F(c) \quad \text{and} \quad f_*(F)(d) = \lim_{c \in \mathcal{C}_{d/}} F(c).$$

Here, the slice categories appearing in the formulas are given by the following two pullbacks:

$$\begin{tikzcd} \mathcal{C}_{/d} \arrow[r] \arrow[d] & \mathcal{D}_{/d} \arrow[d] \\ \mathcal{C} \arrow[r, "f"] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}_{d/} \arrow[r] \arrow[d] & \mathcal{D}_{d/} \arrow[d] \\ \mathcal{C} \arrow[r, "f"] & \mathcal{D} \end{tikzcd}$$

3.17. **Corollary** *Let $(\mathcal{C}, \rho)$ be a quasi-site and $S \in \mathrm{Cov}_\rho(X)$. Let $R \in \mathrm{Sieve}(X)$ and assume that for all objects $Z \in \mathcal{C}$ and $f \in S(Z)$, we have $f^*(R) \in \mathrm{Cov}_\rho(Z)$. Then a ρ-sheaf satisfies descent with respect to R.*

Proof. Consider the subsieve $T \subseteq R$ given by the pullback

$$\begin{tikzcd} T \arrow[r] \arrow[d] & S \arrow[d] \\ R \arrow[r] & y(X) \end{tikzcd}$$

Now given $f \in R(Y)$, i.e. $f\colon y(Y) \to R$, we may consider the diagram of pullback squares

$$\begin{tikzcd} T' \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & S \arrow[d] \\ y(Y) \arrow[r] & R \arrow[r] & y(X) \end{tikzcd}$$

which shows that $T' = f^*(S)$. Since $S \in \mathrm{Cov}_\rho(X)$, the definition of a quasi-topology says that there is then $T'' \subseteq T' = f^*(S)$ with $T'' \in \mathrm{Cov}_\rho(Y)$. By Lemma 3.15 it then suffices that a ρ-sheaf F satisfies descent with respect to T.

To do so, we consider the following: We know that S is the colimit over all representables $y(Z)$ mapping to S, in formulas $S = \operatorname{colim}_{Z\to S} y(Z)$, by an exercise above. Now, since colimits in $\mathrm{PSh}(\mathcal{C})$ are universal, we deduce the isomorphism

$$T \cong \operatorname*{colim}_{Z \xrightarrow{f} S} [y(Z) \times_S T].$$

The diagram

$$\begin{tikzcd} f^*(R) \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & R \arrow[d] \\ y(Z) \arrow[r] & S \arrow[r] & y(X) \end{tikzcd}$$

then shows that

$$T \cong \operatorname*{colim}_{Z \xrightarrow{f} S} f^*(R).$$

With this we can calculate that

$$\begin{aligned} \mathrm{Hom}(S, F) &= \lim_{Z\to S} \mathrm{Hom}(Z, F) \\ &\overset{*}{=} \lim_{Z\to S} \mathrm{Hom}(f^*(R), F) \\ &= \mathrm{Hom}(\operatorname*{colim}_{Z\to S} f^*(R), F) \\ &= \mathrm{Hom}(T, F) \end{aligned}$$

where the equation $*$ uses that by assumption $f^*(R) \in \mathrm{Cov}_\rho(Z)$, so that F satisfies descent with respect to $f^*(R)$. Using that $S \in \mathrm{Cov}_\rho(X)$, we obtain also that the map

$$F(X) \to F(S) = \mathrm{Hom}(S, F)$$

is an isomorphism, so we finally deduce that F satisfies descent with respect to T. $\square$

For a collection $\mathcal{F} = \{F_i\}_{i\in I}$ of presheaves on a small category, we say that a sieve $S \in \mathrm{Sieve}(X)$ is $\mathcal{F}$-descendable, if it is descendable for each presheaf F_i contained in $\mathcal{F}$.

3.18. **Proposition** *Let $\mathcal{C}$ be a small category and $\mathcal{F}$ a collection of presheaves on $\mathcal{C}$. For an object X of $\mathcal{C}$, define*

$$\mathrm{Cov}_{\mathcal{F}}(X) = \{S \in \mathrm{Sieve}(X) \mid \forall f\colon Y \to X, f^*(S) \textit{ is } \mathcal{F}\textit{-descendable}\}.$$

Then the collection $\mathrm{Cov}_{\mathcal{F}}(X)$ of sieves forms a Grothendieck topology on $\mathcal{C}$. Each element of $\mathcal{F}$ is a sheaf with respect to this topology.

Proof. We first check the axioms of a topology. First, we need to see that for all $X \in \mathcal{C}$, we have $y(X) \in \mathrm{Cov}_{\mathcal{F}}(X)$. This means that for all $f\colon Y \to X$, we need to show that $f^*(y(X)) = y(Y)$ is $\mathcal{F}$-descendable, which is true by definition $\mathcal{F}$-descendability.

Second, we need to show that if $S \in \mathrm{Cov}_{\mathcal{F}}(X)$ and $f\colon Y \to X$, then $f^*(S) \in \mathrm{Cov}_{\mathcal{F}}(Y)$. To see this, we need to show that for all $g\colon Z \to Y$, $g^*f^*(S) = (fg)^*(S)$ is $\mathcal{F}$-descendable, which is true by definition of $\mathrm{Cov}_{\mathcal{F}}(X)$.

Third, we need to show that if $S \in \mathrm{Cov}_{\mathcal{F}}(X)$ and $R \in \mathrm{Sieve}(X)$ such that for all $g \in S(Z)$ we have $g^*(R) \in \mathrm{Cov}_{\mathcal{F}}(Z)$, then $R \in \mathrm{Cov}_{\mathcal{F}}(X)$. This means that for all $f\colon Y \to X$, we need to show that $f^*(R)$ is $\mathcal{F}$-descendable. Let us consider the objects $f^*(S) \in \mathrm{Cov}_{\mathcal{F}}(X)$ and $f^*(R) \in \mathrm{Sieve}(Y)$. For all $h \in f^*(S)(Z)$, we have $h^*f^*(R) = (fh)^*(R) \in \mathrm{Cov}_{\mathcal{F}}(Z)$, by assumption. Therefore, by replacing S and R by $f^*(S)$ and $f^*(R)$, it in fact suffices to show that R is $\mathcal{F}$-descendable. But this is what Corollary 3.17 gives: Indeed, we have already verified that the collection $\mathrm{Cov}_{\mathcal{F}}(X)$ forms a quasi-topology and each object of $\mathcal{F}$ is a sheaf with respect to this quasi topology.

We therefore deduce that the collection $\mathrm{Cov}_{\mathcal{F}}(X)$ indeed forms a topology and that, again by construction, each element of $\mathcal{F}$ is a sheaf with respect to this topology. $\square$

Proof of Theorem 3.14. Let F be a τ-sheaf. Since $\mathrm{Cov}_\rho(X) \subseteq \mathrm{Cov}_\tau(X)$ for each object $X \in \mathcal{C}$, we find that F is also a ρ-sheaf. It remains to prove the other implication. So let F be a ρ-sheaf.

Consider the topology τ' on $\mathcal{C}$ given by Proposition 3.18. By construction, τ' contains the sieves contained in ρ, so we find that $\rho \subseteq \langle\rho\rangle = \tau \subseteq \tau'$. Since F is a τ'-sheaf, it is also τ-sheaf as needed. $\square$

Exercise. Show that in the notation of the preceding proof, that the topologies τ and τ' agree.

3.19. **Remark** In light of Theorem 3.14 one may wonder what the role of Axioms (1) and (3) in the definition of a topology play. Clearly, Axiom (1) is not interesting from the point of view of sheaves - any presheaf satisfies descent with respect to the sieve $y(X)$ (by definition). We will now note that the axioms of a topology allow to conclude that the set $\mathrm{Cov}_\tau(X)$ is canonically a cofiltered poset.

We first observe that it is canonically a poset: We have $S \leq S'$ if and only if $S \subseteq S'$ is a subfunctor. Indeed, the category associated to this poset is equivalent to the full subcategory of the slice category $\mathrm{PSh}(\mathcal{C})_{/y(X)}$ on sieves contained in $\mathrm{Cov}_\tau(X)$. Now we claim that this poset is cofiltered. For this, given $S, S' \in \mathrm{Cov}_\tau(X)$, we consider the pullback in presheaves given by $S \times_{y(X)} S'$ (informally, this is the intersection of S and S'). We claim that this pullback is an object of $\mathrm{Cov}_\tau(X)$. Indeed, let us consider the diagram

$$\begin{tikzcd} S'' \arrow[r] \arrow[d] & S \times_{y(X)} S' \arrow[r] \arrow[d] & S' \arrow[d] \\ y(Y) \arrow[r, "f"] & S \arrow[r] & y(X) \end{tikzcd}$$

of pullback squares. Firstly, since $S, S' \in \mathrm{Cov}_\tau(X)$ we find that $S \times_{y(X)} S'$ is a sieve on X. Now, a map $f\colon y(Y) \to S$ is the same datum as an element in $S(Y)$, by the Yoneda lemma. Axiom (2) says that $S'' \in \mathrm{Cov}_\tau(Y)$. Axiom (3) then says that the fact that this is true for any $f \in S(Y)$ suffices to deduce that the sieve $S \times_{y(X)} S'$ is an element of $\mathrm{Cov}_\tau(X)$.

3.20. **Remark** We also remark that the association $X \mapsto \mathrm{Cov}_\tau(X)$ refines to a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$. To see this, for each $f\colon Y \to X$ we have to make show that the map of sets $f^*\colon \mathrm{Cov}_\tau(X) \to \mathrm{Cov}_\tau(Y)$ respects the partial orders on each of these sets: In other words, for $S' \subseteq S$ we have to observe that $f^*S' \subseteq f^*S$. Finally, we note that a presheaf $F \in \mathrm{PSh}(\mathcal{C})$ then gives rise to a functor

$$\mathrm{Cov}_\tau(X)^{\mathrm{op}} \longrightarrow \mathrm{Set} \quad S \mapsto \mathrm{Hom}(S, F) = F(S)$$

whose indexing category is *filtered*. This functor plays a role in the sheafification of presheaves, see Theorem 3.34, and the fact that $\mathrm{Cov}_\tau(X)^{\mathrm{op}}$ is filtered will have the important consequence that the sheafification functor commutes with finite limits - something very special for a left

Bousfield localisation (in fact, this essentially characterises the (∞)-topoi among general presentable (∞)-categories).

3.21. **Example** Let X be a topological space and consider the partially ordered set $\mathrm{Open}(X)$ of open subsets viewed as a category. For $U \in \mathrm{Open}(X)$, we note that a sieve S on U is equivalently described by a collection of subsets of U (namely those $V \subseteq U$ with $S(V) \neq \varnothing$), which is closed under taking open subsets. For an open cover $\{U_i\}_{i\in I}$ of U, we can consider the set of subsets of U which are contained in some U_i. This is a sieve, explicitly given by the following formula:

$$S(V) = \begin{cases} * & \text{if } V \subseteq U_i \text{ for some } i \in I \\ \varnothing & \text{else} \end{cases}$$

We may then define for each $U \in \mathrm{Open}(X)$ the set $\mathrm{Cov}(U)$ as the collection of sieves on U obtained from an open cover of U as just explained. This collection forms a topology on $\mathrm{Open}(X)$, as we recommend to work out as an exercise. Consequently, the category $\mathrm{Open}(X)$ together with the just defined topology forms a site. We will argue below, see Corollary 3.29 and Example 3.33, that a presheaf F on $\mathrm{Open}(X)$ is a sheaf if and only if for all open covers $\{U_i\}_{i\in I}$ of an open set $U \subseteq X$, the canonical map

$$F(U) \to \mathrm{Eq}\big[\prod_{i\in I} F(U_i) \rightrightarrows \prod_{k,l\in I} F(U_k \cap U_l)\big]$$

is a bijection.

Examples of sheaves to keep in mind are

(1) for a topological space X, the association $U \mapsto C(U;Y)$ sending an open to the set of continuous map from U to a fixed topological space Y is a sheaf on X,
(2) for a smooth manifold M, the association $U \mapsto C^\infty(U;N)$ sending an open to smooth functions to another manifold N is a sheaf on the underlying topological space of M,
(3) for a complex manifold X, the association $U \mapsto \mathcal{O}(U;Y)$ sending an open to holomorphic functions to another complex manifold Y is a sheaf on the underlying topological space of X,
(4) for a scheme X, the "structure sheaf" is a sheaf on the underlying topological space of X.

Guided by the above example, we make the following definition.

3.22. **Definition** Let $\mathcal{C}$ be an essentially small category. Let $\mathcal{X} = \{X_i \xrightarrow{p_i} X\}_{i\in I}$ be a family of maps with target X. The sieve $S(\mathcal{X})$ on X generated by the family $\mathcal{X}$ is given by

$$S(\mathcal{X})(Y) = \{Y \xrightarrow{\alpha} X \mid \exists i \in I \text{ such that } \alpha = p_i\alpha_i \text{ for some } \alpha_i \colon Y \to X_i\}.$$

3.23. **Definition** Let $\mathcal{C}$ be an essentially small category. A quasi-covering on $\mathcal{C}$ consists of a set of covering families $\mathcal{X}$ for each object X in $\mathcal{C}$ satisfying the following condition: For any covering family $\mathcal{X} = \{X_i \to X\}_{i\in I}$ of X and any $f\colon Y \to X$, there exists a covering family $\mathcal{Y} = \{Y_j \to Y\}_{j\in J}$ such that the composite $Y_j \to Y \to X$ factors through one of the maps $X_i \to X$, i.e. the composite $Y_j \to X$ lies in $S(\mathcal{X})(Y_j)$.

3.24. **Remark** The definition of a quasi-covering is rigged so that the following is true. Consider for each object X the set $\mathrm{Cov}_\rho(X)$ consisting of the sieves $S(\mathcal{X})$ associated to the covering families elements $\mathcal{X}$. Then the collection of the sets $\mathrm{Cov}_\rho(X)$ forms a quasi-topology on $\mathcal{C}$, the *quasi-topology associated to the quasi-covering*. Indeed, given a covering family $\mathcal{X}$ and a map $f\colon Y \to X$. Let $\mathcal{Y}$ be a covering family as in Definition 3.23. Then we have $S(\mathcal{Y}) \subseteq f^*(S(\mathcal{X}))$.

3.25. **Proposition** *Let $\mathcal{X} = \{X_i \to X\}_{i\in I}$ be a collection of maps with target X. Then in* $\mathrm{PSh}(\mathcal{C})$ *we have the following isomorphism*

$$S(\mathcal{X}) \cong \mathrm{Coeq}\big[\coprod_{k,l\in I} y(X_k) \times_{y(X)} y(X_l) \rightrightarrows \coprod_{i\in I} y(X_i)\big].$$

3.26. **Remark** If $\mathcal{C}$ happens to admit the pullbacks $X_i \times_X X_j$ for all $i, j \in I$ there is a further isomorphism

$$S(\mathcal{X}) \cong \mathrm{Coeq}\big[\coprod_{k,l\in I} y(X_k \times_X X_l) \rightrightarrows \coprod_{i\in I} y(X_i)\big].$$

The maps are given by the two canonical projections of the pullback. This follows from Proposition 3.25 simply because the Yoneda embedding $\mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ preserves limits.

3.27. **Remark** In general, for a covering family $\mathcal{X} = \{X_i \to X\}_{i\in I}$, we can construct a diagram $\check{C}_\bullet(\mathcal{X})\colon \Delta^{\mathrm{op}} \to \mathrm{PSh}(\mathcal{C})$ by setting

$$\check{C}_n(\mathcal{X}) = \coprod_{f\colon [n]\to I} y(X_{f(0)}) \times_{y(X)} \cdots \times_{y(X)} y(X_{f(n)})$$

This simplicial object is called the Cech-nerve associated to the covering family $\mathcal{X}$. In the category of presheaves of anima (and also in the category of presheaves of sets by Lemma 3.2) we also have an isomorphism

$$S(\mathcal{X}) \cong \operatorname*{colim}_{\Delta^{\mathrm{op}}} \check{C}_\bullet(\mathcal{X}).$$

Proof of Proposition 3.25. We recall that $S(\mathcal{X}) \subseteq y(X)$. By construction, we have a canonical map

$$\mathrm{Coeq}\big[\coprod_{k,l\in I} y(X_k) \times_{y(X)} y(X_l) \rightrightarrows \coprod_{i\in I} y(X_i)\big] \to y(X).$$

We wish to show that it is an isomorphism onto $S(\mathcal{X})$. For this, we need to show that the map induces a isomorphism after evaluating at an object $Y \in \mathcal{C}$. Spelling this out, we want to show that there is a bijection

$$\text{(1)} \qquad \mathrm{Coeq}\Big[\coprod_{i,j\in I} \mathrm{Hom}(Y, X_i) \times_{\mathrm{Hom}(Y,X)} \mathrm{Hom}(Y, X_j) \rightrightarrows \coprod_{k\in I} \mathrm{Hom}(Y, X_k)\Big] \to S(\mathcal{X})(Y).$$

For this we recall that

$$S(\mathcal{X})(Y) = \{Y \xrightarrow{\alpha} X \mid \exists i \in I \text{ such that } \alpha = p_i\alpha_i \text{ for some } i \in I\}.$$

Now, for each $\alpha \in S(\mathcal{X})(Y)$, choosing an $\alpha_i \in \mathrm{Hom}(Y, X_i)$ shows that the map (1) is surjective. Now assume given two maps $\alpha_k \colon Y \to X_k$ and $\alpha_l \colon Y \to X_l$ with the same image under the map in question. This simply means that $p_k\alpha_k = p_l\alpha_l \colon Y \to X$, which in turn means that (α_k, α_l) give a point in $\mathrm{Hom}(Y, X_k)\times_{\mathrm{Hom}(Y,X)}\mathrm{Hom}(Y, X_l)$ mapping to α_k and α_l under the two canonical projections. This, however, implies that α_k and α_l are identified in the coequaliser, so the proposition is shown. $\square$

3.28. **Remark** For a general sieve $R \in \mathrm{Sieve}(X)$, there is still a presentation of R as a coequaliser of coproducts of representables: This is a consequence of the presentation of *any* presheaf F as the colimit of representables mapping to F, and the general way of writing colimits as coequalisers of coproducts. The formula that comes out is:

$$R \cong \mathrm{Coeq}\Big[\coprod_{Y\in\mathcal{C}, f\in R(Y), Z\xrightarrow{g}Y} y(Z) \rightrightarrows \coprod_{Y\in\mathcal{C}, f\in R(Y)} y(Y)\Big].$$

I don't know yet whether this presentation will be useful for us later.

3.29. **Corollary** *Let F be a presheaf on $\mathcal{C}$ and $\mathcal{X} = \{X_i \to X\}_{i\in I}$ a family, and suppose the fibre products $X_i \times_X X_j$ exist in $\mathcal{C}$. Then F satisfies descent with respect to the sieve $S(\mathcal{X})$ generated by $\mathcal{X}$ if and only if the canonical map*

$$F(X) \to \mathrm{Eq}\big[\prod_{i\in i} F(X_i) \rightrightarrows \prod_{k,l\in I} F(X_k \times_X X_l)\big]$$

is a bijection.

Proof. This is a consequence of Proposition 3.25, Remark 3.26 and the Yoneda lemma. $\square$

We add a further notion of coverings for completeness (mostly):

3.30. **Definition** Let $\mathcal{C}$ be an essentially small category. A covering on $\mathcal{C}$ consists of collections of covering families $\mathcal{X} = \{X_i \to X\}_{i\in I}$ for each object X in $\mathcal{C}$ such that the following axioms hold:

(1) For an isomorphism $f \colon X' \to X$, we have that $\{f\}$ is a covering family.
(2) For every covering family $\mathcal{X} = \{X_i \to X\}_{i\in I}$ and every map $Y \to X$, the pullbacks $Y_i = Y \times_X X_i$ exist and $\mathcal{Y} \overset{\text{def}}{=} Y \times_X \mathcal{X} = \{Y_i \to Y\}_{i\in I}$ is a covering family.

(3) Given a covering family $\mathcal{X} = \{X_i \to X\}_{i \in I}$ and for each $i \in I$, we have a further covering family $\mathcal{Y}_i = \{Y_{i,j} \to X_i\}_{j \in J(i)}$, then the collection $\{Y_{i,j} \to X\}_{i \in I, j \in J(i)}$ forms another covering family of X.

3.31. **Remark** A covering is in particular a quasi-covering satisfying more axioms: Indeed, axiom (2) guarantees that a covering is a quasi-covering.

One might wonder whether a covering is a quasi-covering whose associated quasi-topology, see Remark 3.24, is a topology. This is not quite true, but almost. Let us (redefine)

$$\mathrm{Cov}(X) = \{S \in \mathrm{Sieve}(X) \mid \exists \text{ a covering family } \mathcal{X} \text{ s.t. } S(\mathcal{X}) \subseteq S\}$$

We claim that this defines a topology on $\mathcal{C}$ and this topology coincides with the one generated by the quasi-topology associated to the underlying quasi-covering of the covering. We will now argue that this is in fact a topology and leave the other claim as an exercise.

Axiom (1) of a topology follows from the fact that $\{\mathrm{id}_X : X \to X\}$ is a covering family. For axiom (2), let $S \in \mathrm{Cov}(X)$, and let $\mathcal{X}$ be a covering family with $S(\mathcal{X}) \subseteq S$. Then $f^*(S(\mathcal{X})) \subseteq f^*(S)$, so it suffices to note that

$$f^*(S(\mathcal{X})) = S(f^*(\mathcal{X})) \overset{\text{def}}{=} S(Y \times_X \mathcal{X}).$$

For axiom (3), suppose given $S \in \mathrm{Cov}(X)$ and $R \in \mathrm{Sieve}(X)$ and assume that for each Y in $\mathcal{C}$ and $f \in S(Y)$, we have $f^*(R) \in \mathrm{Cov}(Y)$. We then have to show that $R \in \mathrm{Cov}(X)$. By assumption we have $S(\mathcal{X}) \subseteq S$ for some covering family of X. If $\mathcal{X}$ is the empty family, then $S(\mathcal{X}) = \varnothing \subseteq R$ and we are done. If $\mathcal{X}$ is not the empty family, and writing $\mathcal{X} = \{X_i \overset{p_i}{\to} X\}_{i \in I}$ we have $p_i \in S(\mathcal{X})(X_i) \subseteq S(X_i)$. By assumption, there exists a covering family $\mathcal{Y}_i = \{Y_{i,j} \to X_i\}_{j \in J(i)}$ such that $S(\mathcal{Y}_i) \subseteq p_i^*(R)$. By axiom 3 of coverings, we may form the new covering $\bar{\mathcal{X}} = \{Y_{i,j} \to X_i \to X\}_{i \in I, j \in J(i)}$. Then for all $i \in I$ and $j \in J(i)$, we have $Y_{i,j} \to X_i \to X$ is an element of $R(Y_{i,j})$. We deduce that $S(\bar{\mathcal{X}}) \subseteq R$ which was to be shown.

3.32. **Corollary** *Let $\mathcal{C}$ be a category equipped with a quasi-covering where the pullbacks appearing in Remark 3.26 exist (for instance if the quasi-covering is a covering) and let $(\mathcal{C}, \tau)$ be site associated to the quasi-site obtained from the quasi-covering as in Remark 3.24. Then a presheaf F is a τ-sheaf if and only if for every covering family $\mathcal{X}$ of an object X, the canonical map*

$$F(X) \to \mathrm{Eq}\big[\prod_{i \in i} F(X_i) \rightrightarrows \prod_{k,l \in I} F(X_k \times_X X_l)\big]$$

is a bijection.

Proof. Combine Theorem 3.14 with Corollary 3.29. □

3.33. **Example** Let X be a topological space. Consider the quasi-covering given by open covers: For $U \in \mathrm{Open}(X)$, the covering families are families $\{U_i \subseteq U\}_{i \in I}$ which cover U. It is a good exercise to check that this is in fact a covering in the sense of Definition 3.30. Its associated topology is the one concretely described in Example 3.21. In particular, a presheaf F on X is a sheaf if and only if for each open cover $\{U_i\}$ of an open subset U of X, the canonical map

$$F(U) \to \mathrm{Eq}\big[\prod_{i \in I} F(U_i) \rightrightarrows \prod_{k,l \in I} F(U_k \cap U_l)\big]$$

is a bijection.

We finish this section with a discussion of sheafification, though we will not give full proofs.

3.34. **Theorem** *Let* $(\mathcal{C}, \tau)$ *be a site. Then the inclusion* $\mathrm{Sh}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ *admits a left adjoint called* sheafification. *The sheafification functor commutes with finite limits.*

Proof. Let F be a presheaf on $\mathcal{C}$. We define a new presheaf $F^\dagger$ on $\mathcal{C}$ as follows. We recall Remark 3.20 and set

$$F^\dagger(X) = \operatorname*{colim}_{S \in \mathrm{Cov}_\tau(X)^{\mathrm{op}}} F(S).$$

One can see that this canonically refines to a presheaf $F^\dagger$. Moreover, there is a canonical map $F \to F^\dagger$, induced by the maps $S \to y(X)$ for $S \in \mathrm{Cov}_\tau(X)$.

The first thing one then shows is that $F^\dagger$ is always a τ-*separated* presheaf: For any $S \in \mathrm{Cov}_\tau(X)$, we have that the map $F(X) \to F(S)$ is injective. Second, one shows that if F is a τ-separated presheaf, then $F^\dagger$ is a sheaf. Consequently, $(F^\dagger)^\dagger$ is a sheaf, and it receives a canonical map from F. We claim that the association $F \mapsto (F^\dagger)^\dagger$ refines to a left adjoint of the inclusion of sheaves into presheaves. We will not verify these claims here, see for instance [**?**, Tag 7.10] in the case where the topology is generated by a covering.

We will however, show that the so constructed sheafification functor preserves finite limits. So let $F\colon I \to \mathrm{PSh}(\mathcal{C})$ be a finite diagram (e.g. the equalisers or a finite products - by Lemma 1.16 it suffices to consider these cases) of presheaves and let $\mathcal{F} = \lim_I F(i)$. Then we have

$$\begin{aligned}
\mathcal{F}^\dagger(X) &= \operatorname*{colim}_{S \in \mathrm{Cov}_\tau(X)} \mathcal{F}(S) \\
&= \operatorname*{colim}_{S \in \mathrm{Cov}_\tau(X)} \lim_{i \in I} F(i)(S) \\
&\overset{(*)}{=} \lim_{i \in I} \operatorname*{colim}_{S \in \mathrm{Cov}_\tau(X)} F(i)(S) \\
&= \lim_{i \in I} F(i)^\dagger(X)
\end{aligned}$$

where equality $(*)$ holds since the colimit in question is filtered, and in the category of sets, filtered colimits commute with finite limits. Applying this argument twice, we deduce that the sheafification functor commutes with finite limits. □

3.35. **Remark** The first part of Theorem 3.34, i.e. that the inclusion $\mathrm{Sh}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ admits a left adjoint is a special case of the adjoint functor theorem. Namely, F is a sheaf if and only if it is *local* with respect to the *set* of maps

$$\{S \to y(X) \mid X \in \mathcal{C} \text{ and } S \in \mathrm{Cov}(X)\}$$

which we call *generating local equivalences.* The theory of presentable (∞)-categories then shows that the inclusion $\mathrm{Sh}(\mathcal{C})$ admits a left adjoint L. A morphism in $\mathrm{PSh}(\mathcal{C})$ is then called a *local equivalence* or an L-*equivalence.* Moreover, the left adjoint L is left exact (i.e. preserves finite limits) if and only if the local equivalences are stable under pullbacks, see [**?**, Lemma 6.2.1.1]. In our case, axiom (2) of a topology implies this for the generating local equivalences: Indeed, the axiom says that for a morphism $f\colon Y \to X$ and $S \in \mathrm{Cov}(X)$, the left vertical map in the pullback diagram

$$\begin{tikzcd}
f^*(S) \arrow[r] \arrow[d] & S \arrow[d] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

is again a (generating) local equivalence. However, there are more local equivalences than generating local equivalences, and it is not obvious that the pullback of a general local equivalence is again a local equivalence. Therefore, some further argument is required to show that sheafification is left exact, but surely, the above suggests that this should be true.

3.36. **Remark** Everything we have done in this section remains true when we consider presheaves and sheaves on a site $\mathcal{C}$ which take values in a category $\mathcal{D}$ which comes with a conservative, faithful, and limit- and filtered colimit preserving functor to Set. For instance, for any algebraically looking category like Ring, Ab, $\mathrm{Mod}(R)$, etc.

Indeed, in any of these cases, a presheaf with values in $\mathcal{D}$ is a sheaf if and only if its underlying set valued presheaf is a sheaf. Likewise, the sheafification of the set-valued presheaf underlying a $\mathcal{D}$-valued presheaf is canonically $\mathcal{D}$-valued and consequently a sheaf when viewed as such. In other words, the diagrams

$$\begin{tikzcd} \mathrm{Sh}(\mathcal{C};\mathcal{D}) \arrow[r, hook] \arrow[d] & \mathrm{PSh}(\mathcal{C};\mathcal{D}) \arrow[r, "L"] \arrow[d] & \mathrm{Sh}(\mathcal{C};\mathcal{D}) \arrow[d] \\ \mathrm{Sh}(\mathcal{C}) \arrow[r, hook] & \mathrm{PSh}(\mathcal{C}) \arrow[r, "L"] & \mathrm{Sh}(\mathcal{C}) \end{tikzcd}$$

commute, where the horizontal maps are the inclusion of sheaves in presheaves and the sheafification functors, and the vertical maps are induced by the functor $\mathcal{D} \to \mathrm{Set}$.

4. Condensed sets

To make formal sense, let us fix an uncountable strong limit cardinal κ (this implies that for any $\lambda < \kappa$, we have $2^\lambda < \kappa$) - the existence of such cardinals can be shown without further set theoretic assumption (i.e. we need not assume any large cardinal axioms as we did in the very first lecture). In addition, we note here without proof that for any cardinal τ, there exists $\kappa \geq \tau$ with κ an uncountable strong limit cardinal.

Let CH_κ denote the full subcategory of compact Hausdorff spaces which are κ-small, i.e. have cardinality less than κ, and similarly we write $\mathrm{tdCH}_\kappa = \mathrm{tdCH} \cap \mathrm{CH}_\kappa$, and $\mathrm{edCH}_\kappa = \mathrm{edCH} \cap \mathrm{CH}_\kappa$.

We will define quasi-coverings on the categories

$$\mathrm{edCH}_\kappa \subseteq \mathrm{tdCH}_\kappa \subseteq \mathrm{CH}_\kappa$$

and then obtain κ-condensed sets as the category of sheaves on either of these categories, as we will show. This is the reason why we need to pay some attention to κ's (i.e. set theory): Technically speaking, we need to consider an essentially small category on which we consider sheaves, but the category CH of compact Hausdorff spaces is not essentially small. However, its subcategory CH_κ is, and it is so for each κ. Later, we will also show what happens when we change from κ to a bigger κ'.

4.1. **Definition** Let X be an object of any of the above three categories. A family $\mathcal{X} = \{X_i \xrightarrow{p_i} X\}_{i\in I}$ with I finite is called of type

(1) if the map $\coprod\limits_{i\in I} X_i \to X$ is surjective,
(2) if the map $\coprod\limits_{i\in I} X_i \to X$ is bijective, and
(3) if $\mathcal{X} = \{X' \xrightarrow{p} X\}$ with p surjective.

4.2. **Remark** We recall from Corollary 2.35 and Lemma 2.34 that in the above definition, the word "surjective" may equivalently be replaced by "quotient map" and the word "bijective" may equivalently be replaced by "homeomorphism".

4.3. **Lemma** *The collection of families of type (1) form a covering on* CH_κ *and* tdCH_κ *and a quasi-covering on* edCH_κ*. The collection of families of type (2) or (3) form a quasi-covering on* CH_κ*,* tdCH_κ*, and* edCH_κ*. In addition, the families of type (1) and the families of type (2) or (3) generate the same topology, the* condensed topology*, on each of the categories* CH_κ*,* tdCH_κ*, and* edCH_κ*.*

Proof. The categories CH_κ and tdCH_κ have all κ-small limits, and in particular all pullbacks. It follows immediately from the axioms that the families of type (1) form a covering on CH_κ and tdCH_κ.

The families of type (2) or (3) satisfy axioms (1) and (2) of a covering (in particular, these families form a quasi-covering), but not axiom (3): In fact, given any covering on CH_κ or tdCH_κ which contains the families of type (2) and (3), it also contains the families of type (1): Indeed, let $\mathcal{X} = \{X_i \to X\}_{i\in I}$ be a family of type (1). Then $\{X_i \to \amalg_{i\in I} X_i\}_{i\in I}$ is a family of type (2). In addition $\{\amalg_{i\in I} X_i \to X\}$ is a family of type (3). Axiom (3) of a covering then says that also the family $\{X_i \to \amalg_{i\in I} X_i \to X\}_{i\in I}$ is a covering family.

Next we show that the families of type (1) as well as those of type (2) or (3) form a quasi-covering of edCH_κ. For this, let $\mathcal{X} = \{X_i \to X\}_{i\in I}$ a family of some type and $Y \to X$ a map in edCH_κ. Consider the pullback $Y_i = Y \times_X X_i$ in CH. Typically, this pullback is not contained in edCH_κ, but see the exercise below. We have seen earlier that for each $i \in I$, there are surjections $Y_i' \to Y_i$ with $Y_i' \in \mathrm{edCH}$; at this point in addition we need to take care of cardinalities: We need to show that there is Y_i' of cardinality $< \kappa$. But the earlier argument showed that we may choose $Y_i' = \beta(Y_i^\delta)$ and we have seen that $|\beta(M^\delta)| < 2^{2^{|M|}}$ which is smaller than κ since κ is a strong limit cardinal. The family $\{Y_i' \to Y_i\}_{i\in I}$ is then again of the same type as the family $\mathcal{X} = \{X_i \to X\}_{i\in I}$. Since, by construction, the maps $Y_i' \to Y_i \to X$ factor through the map $X_i \to X$, the claimed families indeed form quasi-coverings on edCH_κ.

It finally remains to show that the topology generated by the families of type (1) and the topology generated by the families of type (2) or (3) coincide on each of the categories CH_κ, tdCH_κ, and edCH_κ. Let τ be the topology generated by families of type (1) and τ' be the topology generated by families of type (2) or (3). We need to show that $\tau = \tau'$. We have $\tau' \subseteq \tau$ since families of type (2) and (3) are also of type (1). It remains to show that $\tau' \subseteq \tau$. For this, it suffices to show that τ' contains $S(\mathcal{X})$ for $\mathcal{X} = \{X_i \xrightarrow{p_i} X\}$ a family of type (1). Let $p \colon \amalg_{i\in I} X_i \to X$ be the map induced by the p_i's. Then $S(\{p\}) \in \mathrm{Cov}_{\tau'}(X)$. By axiom (3) of a topology, it suffices to show that for each $f \in S(\{p\})(Y)$, we have $f^*(S(\mathcal{X})) \in \mathrm{Cov}_{\tau'}(X)$. Since $f \in S(\{p\})(Y)$, we can write f as a composite

$$Y \xrightarrow{g} \amalg_{i\in I} X_i \xrightarrow{p} X.$$

Then we have $f^*(S(\mathcal{X})) = g^*p^*(S(\mathcal{X}))$, so by axiom (2) of a topology, it suffices to show that $p^*(S(\mathcal{X})) \in \mathrm{Cov}_{\tau'}(\amalg_{i\in I} X_i)$. But we have

$$p^*(S(\mathcal{X})) = S(\{X_i \to \amalg_I X_i\}_{i\in I})$$

as follows from the fact that the diagram

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] & X_i \arrow[d, "p_i"] \\ \amalg_{i\in I} X_i \arrow[r, "p"] & X \end{tikzcd}$$

is a pullback. Now, the family $\{X_i \to \amalg_{i\in I} X_i\}_{i\in I}$ is a family of type (2) and hence its associated sieve is contained in $\mathrm{Cov}_{\tau'}(\amalg_{i\in I} X_i)$. This concludes the proof of the lemma. □

Exercise. Let $\mathcal{X} = \{X_i \to X\}$ be a family of type (2) in edCH_κ and $f \colon Y \to X$ a map in edCH_κ. Show that the pullbacks $Y_i = Y \times_X X_i$ exist (in edCH_κ) and that the resulting family $\mathcal{Y} = \{Y_i \to Y\}$ again forms a family of type (2).

4.4. **Corollary** *A presheaf on* $\mathrm{CH}_\kappa, \mathrm{tdCH}_\kappa, \mathrm{edCH}_\kappa$ *is a sheaf for the condensed topology if and only if it satisfies descent with respect to the sieves generated by families of type (2) or (3).*

Proof. This is a consequence of Lemma 4.3 and Corollary 3.32. □

4.5. **Theorem** *The inclusions* $\mathrm{edCH}_\kappa \subseteq \mathrm{tdCH}_\kappa \subseteq \mathrm{CH}_\kappa$ *induce equivalences of categories*

$$\mathrm{Sh}(\mathrm{CH}_\kappa) \xrightarrow{\simeq} \mathrm{Sh}(\mathrm{tdCH}_\kappa) \xrightarrow{\simeq} \mathrm{Sh}(\mathrm{edCH}_\kappa).$$

Proof. Let us denote by $\mathcal{C}_\kappa$ either of the two categories $\mathrm{tdCH}_\kappa, \mathrm{CH}_\kappa$. We denote by $i \colon \mathrm{edCH}_\kappa \to \mathcal{C}_\kappa$ the fully faithful inclusion. We first note that we have an adjunction

$$i^* \colon \mathrm{PSh}(\mathrm{edCH}_\kappa) \rightleftarrows \mathrm{PSh}(\mathcal{C}_\kappa) \colon i_*$$

with i_* the right adjoint of i^* and hence given by right Kan extension. Firstly, since i is fully faithful, so is i_*. We now claim that both of the two functors restrict to the dashed arrows in the diagram

$$\begin{tikzcd} \mathrm{Sh}(\mathcal{C}_\kappa) \arrow[r, dashed, shift left, "i^*"] \arrow[d, hook] & \mathrm{Sh}(\mathrm{edCH}_\kappa) \arrow[l, dashed, shift left, "i_*"] \arrow[d, hook] \\ \mathrm{PSh}(\mathcal{C}_\kappa) \arrow[r, shift left, "i^*"] & \mathrm{PSh}(\mathrm{edCH}_\kappa) \arrow[l, shift left, "i_*"] \end{tikzcd}$$

Here, the vertical maps are the inclusions of sheaves into presheaves. We begin with i^*. So let $F \in \mathrm{Sh}(\mathcal{C}_\kappa)$ and we wish to show that $i^*(F)$ is a sheaf on edCH_κ. So let $X \in \mathrm{edCH}_\kappa$. By Corollary 4.4, we need to show that $i^*(F)$ satisfies descent with respect to a covering sieve associated to a family of type (2) or (3). But any such family is also such a family when we view X as an object of $\mathcal{C}_\kappa$. Therefore, $i^*(F)$ satisfies descent since F is a sheaf on $\mathcal{C}$.

Next, we now show that i_* preserves sheaves. So let $F \in \mathrm{Sh}(\mathrm{edCH}_\kappa)$, and $S \in \mathrm{Cov}(X)$ for some object $X \in \mathcal{C}$. We wish to show that the map

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}_\kappa)}(X, i_*(F)) \longrightarrow \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}_\kappa)}(S, i_*(F))$$

is an isomorphism, i.e. that $i_*(F)$ satisfies descent with respect to the sieve S. By adjunction, this map is isomorphic to

$$\mathrm{Hom}_{\mathrm{PSh}(\mathrm{edCH}_\kappa)}(i^*(X), F) \longrightarrow \mathrm{Hom}_{\mathrm{PSh}(\mathrm{edCH}_\kappa)}(i^*(S), \mathcal{F})$$

and so it suffices to show that $i^*(S) \to i^*(X)$ is an isomorphism after sheafification (i.e. after applying the sheafification functor L). We now note that in general, a map $G \to H$ of presheaves on any site $\mathcal{D}$ is an L-equivalence if for all objects $T \in \mathcal{D}$, and all maps $y(T) \to H$, the pulled back map $G \times_H y(T) \to y(T)$ is an L-equivalence. Indeed, we know that

$$H \cong \operatorname*{colim}_{T \to H} y(T).$$

Furthermore, colimits in $\mathrm{PSh}(\mathcal{C})$ are universal, so that

$$G \cong \operatorname*{colim}_{T \to H} \big[G \times_H y(T)\big].$$

Similarly, the map from $G \to H$ also identifies with the colimit of the maps $G \times_H y(T) \to y(T)$. In total we obtain

$$\begin{aligned} L(G \to H) &= L(\operatorname*{colim}_{T \to H} \big[G \times_H y(T) \to y(T)\big]) \\ &= L(\operatorname*{colim}_{T \to H} \big[L(G \times_H y(T) \to y(T))\big]) \end{aligned}$$

where in the second line we have used that L, viewed as a functor $\mathrm{PSh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{D})$ preserves colimits. Coming back to the situation at hand, it therefore suffices to show that for each $T \in \mathrm{edCH}_\kappa$ and every map $f\colon y(T) \to i^*(X)$, i.e. any map $T \to X$ in $\mathcal{C}$, the map $i^*(S) \times_{i^*(X)} y(T) \to y(T)$ is an L-equivalence in $\mathrm{PSh}(\mathcal{C}_\kappa)$ - equivalently we need to show that any sheaf F on edCH_κ satisfies descent with respect to $i^*(S) \times_{i^*(X)} y(T)$. Now, by Corollary 4.4, we may assume that $S = S(\mathcal{X})$ for a family $\mathcal{X} = \{X_i \to X\}_{i \in I}$ of type (2) or (3). Written suggestively, we then have

$$i^*(S) \times_{i^*(X)} y(T) = S(\{X_i \times_X T \to T\})$$

with the caveat that the fibre products appearing need not in general be extremally disconnected, and so this is not literally the sieve generated by a covering family on edCH_κ. This is true, however, for families of type (2) by an exercise above. Hence, we must now consider the case where $\mathcal{X}$ is a family of type (3), i.e. where $\mathcal{X} = \{Y \to X\}$ consists of a single surjection. In this case, we have

$$i^*(S) \times_{i^*(X)} y(T) = S(\{T \times_X Y \to T\}$$

where $T \times_X Y \to T$ is a surjection with T extremally disconnected. It follows that this surjection admits a section and hence that

$$S(\{T \times_X Y \to T\} = y(T)$$

so that F tautologically has descent with respect to it.

We have now shown that the adjunction (i^*, i_*) on presheaves restricts to an adjunction

$$i^* \colon \mathrm{Sh}(\mathcal{C}_\kappa) \leftrightarrows \mathrm{Sh}(\mathrm{edCH}_\kappa) \colon i_*$$

on the level of sheaves. Moreover, i_* on the level of sheaves is fully faithful because i_* on the level of presheaves is fully faithful. To conclude the theorem, it finally suffices to show that i^* is conservative. While not true on the level of presheaves, it becomes true when restricted to sheaves: Let $F \to G$ be a map in $\mathrm{Sh}(\mathcal{C}_\kappa)$ and assume that the induced map $i^*(F) \to i^*(G)$ is an isomorphism. We then need to show that for each $X \in \mathcal{C}_\kappa$, the map $F(X) \to G(X)$ is an isomorphism. Choose a surjection $X' \to X$ with X' extremally disconnected and consider the diagram

$$\begin{tikzcd}
F(X) \arrow[r] \arrow[d] & F(X') \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & F(X' \times_X X') \arrow[r, hook] \arrow[d] & F(X'') \arrow[d, "\cong"] \\
G(X) \arrow[r] & G(X') \arrow[r, shift left] \arrow[r, shift right] & G(X' \times_X X') \arrow[r, hook] & G(X'')
\end{tikzcd}$$

where $X'' \to X' \times_X X'$ is a further surjection with X'' extremally disconnected. Note again that these surjections can be chosen within κ-small spaces as we have also observed earlier. Since F is a sheaf, the first three terms form a horizontal equaliser diagram, and likewise for G. Using again that F and G are sheaves, the right most horizontal maps are injective. We conclude that $F(X) \to G(X)$ is an isomorphism, as needed. □

4.6. **Definition** A κ-condensed set is an object of any of the above three (equivalent) categories. For definiteness, we view a κ-condensed set as a sheaf on κ-small profinite spaces (i.e. on tdCH_κ). We will write $\mathrm{Cond}_\kappa(\mathrm{Set})$ for the category of κ-condensed sets.

4.7. **Remark** Let $\mathcal{D}$ be a category equipped with a conservative, faithful, limit and filtered colimit preserving functor to Set. Then we also have that the canonical functors

$$\mathrm{Sh}(\mathrm{CH}_\kappa; \mathcal{D}) \to \mathrm{Sh}(\mathrm{tdCH}_\kappa; \mathcal{D}) \to \mathrm{Sh}(\mathrm{edCH}_\kappa; \mathcal{D})$$

are equivalences. For such a category $\mathcal{D}$, we denote the resulting single category of sheaves by $\mathrm{Cond}_\kappa(\mathcal{D})$. For $\mathcal{D} = \mathrm{Ab}, \mathrm{Ring}, \mathrm{Mod}(R)$, one has canonical equivalences

$$\mathrm{Cond}_\kappa(\mathrm{Ab}) = \mathrm{Ab}(\mathrm{Cond}_\kappa(\mathrm{Set})), \quad \mathrm{Cond}_\kappa(\mathrm{Ring}) = \mathrm{Ring}(\mathrm{Cond}_\kappa(\mathrm{Set})), \text{ etc.}$$

Here, to form for instance $\mathrm{Ab}(\mathrm{Cond}_\kappa(\mathrm{Set}))$ we note that $\mathrm{Cond}_\kappa(\mathrm{Set})$ has finite products and so we may form abelian group objects with respect to the cartesian symmetric monoidal structure. Concretely, these are $X \in \mathrm{Cond}_\kappa(\mathrm{Set})$, together with maps $* \to X$ and $X \times X \to X$ of condensed sets satisfying the usual unitality, associativity and commutativity constraints, for which the shear map is an isomorphism (or equivalently, where the values $X(T)$ for $T \in \mathrm{tdCH}_\kappa$ are abelian groups).

4.8. **Proposition** *The canonical functor*

$$\mathrm{Cond}_\kappa(\mathrm{Set}) \to \mathrm{Fun}^\times(\mathrm{edCH}_\kappa^{\mathrm{op}}, \mathrm{Set})$$

is an equivalence of categories. Here, the superscript $\times$ *refers to functors* X *which preserve finite products, or equivalently where the canonical maps*

(1) $X(\emptyset) \to *$, *and*
(2) $X(T \amalg T') \to X(T) \times X(T')$

are isomorphisms for all $T, T' \in \mathrm{edCH}_\kappa$.

4.9. **Remark** Again, by passing to abelian group objects or ring objects, the same equivalence holds true for condensed abelian groups or condensed rings. In fact, the same equivalence holds true for $\mathrm{Cond}_\kappa(\mathcal{D})$ under the usual assumptions on $\mathcal{D}$.

Proof of Proposition 4.8. That the two descriptions of product preserving functors are equivalent is a simple induction argument. To see that the functor is well-defined, we first show that a presheaf X on extremally disconnected spaces satisfies descent with respect to type (2) families if and only if X preserves finite products: Indeed, let $\mathcal{T} = \{T_i \to T\}_{i \in I}$ be a family of type (2), so without loss of generality we have $T = \amalg_{i \in I} T_i$. Since the required pullbacks exist in this case, we deduce from Remark 3.26 that X satisfies descent if and only if the map

$$X(T) \to \mathrm{Eq}\big[\prod_{i \in I} X(T_i) \rightrightarrows \prod_{k,l \in I} X(T_k \times_T T_l)\big]$$

is an isomorphism. Now, for $k \neq l$, we have $T_k \times_T T_l = \varnothing$ and $T_k \times_T T_k = T_k$. Consequently, we deduce that X satisfies descent if and only if the map

$$X(T) \to \mathrm{Eq}\big[\prod_{i \in I} X(T_i) \rightrightarrows \prod_{i \in I} X(T_i) \times \prod_{k \neq l} X(\varnothing)\big]$$

is an isomorphism. Now, using the empty covering, we know that if X satisfies descent, then $X(\varnothing) = *$. Under this assumption, the above equaliser is given by $\prod_i X(T_i)$ since the two maps in the equaliser agree. Taken alltogether, we may deduce that X satisfies descent with respect to type (2) families if and only if X preserves finite products.

In particular, we have shown that the functor under investigation is well-defined and fully faithful since both sides are compatibly full subcategories of $\mathrm{PSh}(\mathrm{edCH}_\kappa)$. It then suffices to show that any presheaf X satisfies descent with respect to type (3) families. So let us consider a family $\{T' \xrightarrow{p} T\}$ of a single surjection between extremally disconnected compact Hausdorff spaces. From Theorem 2.53 we know that p admits a section. Therefore, we find

$$S(\mathcal{T})(Y) = \{Y \to T \mid \exists \text{ factorisation over } p\} = \mathrm{Hom}(Y, T)$$

so that X tautologically satisfies descent with respect to $S(\mathcal{T})$. $\square$

Exercise. Let $p\colon X' \to X$ be a map with a section $s\colon X \to X'$ in some category $\mathcal{C}$ which admits pullbacks. Let F be a presheaf on $\mathcal{C}$. Show (by hand) that the canonical diagram

$$F(X) \to F(X') \rightrightarrows F(X' \times_X X')$$

is an equaliser diagram.

4.10. **Definition** The *underlying set* functor $\mathrm{Cond}_\kappa(\mathrm{Set}) \to \mathrm{Set}$ is given by $X \mapsto X(*)$.

For the following, we recall that a topological space X is κ-compactly generated if a map $X \to Y$ is continuous if for all $T \in \mathrm{CH}_\kappa$ and all maps $T \to X$, the composite $T \to X \to Y$ is continuous.

4.11. **Theorem** *There is a canonical functor* $\underline{(-)}\colon \mathrm{Top} \to \mathrm{Cond}_\kappa(\mathrm{Set})$. *This functor is faithful, and full when restricted to the full subcategory of κ-compactly generated spaces.*

Proof. The functor is the restricted Yoneda embedding: It sends $X \in \mathrm{Top}$ to the κ-condensed set $\underline{X}$ with $\underline{X}(T) = \mathrm{Map}(T, X) = \mathrm{Hom}_{\mathrm{Top}}(T, X)$ where $T \in \mathrm{edCH}_\kappa$. We observe that the composite

$$\mathrm{Top} \to \mathrm{Cond}(\mathrm{Set}) \to \mathrm{Set}$$

is the canonical forgetful functor that forgets the topology of a topological space. Since this composite functor is faithful, so is the first functor. To see the fullness, let X and Y be κ-compactly generated spaces and let $f\colon \underline{X} \to \underline{Y}$ be a map of the associated condensed sets. Then the induced map $f(*)\colon \underline{X}(*) \to \underline{Y}(*)$ is a map $X \to Y$. Consider the map $f(T)\colon \underline{X}(T) \to \underline{Y}(T)$ and let $t\colon * \to T$ pick out a point $t \in T$. Then, by naturality, the diagram

$$\begin{tikzcd} \mathrm{Map}(T,X) \arrow[r, "f(T)"] \arrow[d, "t^*"] & \mathrm{Map}(T,Y) \arrow[d, "t^*"] \\ X \arrow[r, "f(*)"] & Y \end{tikzcd}$$

commutes, showing that $f(T)$ is given by post composition with the map $f(*)$. It therefore suffices to show that the map $f(*)$ is continuous. To show this, since X is κ-compactly generated, it suffices to show that for each continuous map $\alpha\colon T \to X$, the composite $T \to X \to Y$ is continuous. But we have just argued that this composite is the image of α under the map $f(T)$, and is therefore an element of $Y(T) = \mathrm{Map}(T, Y)$, the continuous maps from T to Y. $\square$

4.12. **Proposition** *The functor* $\mathrm{Top} \to \mathrm{Cond}_\kappa(\mathrm{Set})$ *admits a left adjoint denoted by* $X \mapsto X(*)_{\mathrm{top}}$.

Proof. The left adjoint is given as follows: The underlying set is given by $X(*)$, and the topology on $X(*)$ is the quotient topology coming from the surjection

$$\coprod_{S \to X} S \to X(*)$$

where S runs through all objects of tdCH_κ equipped with maps to X (i.e. points of $X(S)$. We denote the resulting topological space by $X(*)_{\mathrm{top}}$. To see that this is indeed a left adjoint, we consider $X \in \mathrm{Cond}_\kappa(\mathrm{Set})$ and $A \in \mathrm{Top}$. Then we need to show that there is a natural bijection

$$\mathrm{Hom}_{\mathrm{Cond}_\kappa(\mathrm{Set})}(X, \underline{A}) \overset{\cong}{\to} \mathrm{Hom}_{\mathrm{Top}}(X(*)_{\mathrm{top}}, A).$$

We claim that there is a canonical map from left to right, which is the bijection we are looking for. Indeed, the map is obtained by evaluating a map $X \to \underline{A}$ on $*$, which a priori provides a map of sets. We have to show that this map of sets is continuous. To do so, it suffices to show that for any $S \in \mathrm{tdCH}_\kappa$ and map $S \to X$, the composite $S \to X(*) \to A$ is continuous. By construction, it is obtained from the map $\underline{S} \to X \to \underline{A}$ of κ-condensed sets by evaluating on $*$. We have argued in Theorem 4.11 that this map is continuous since S is (by definition) κ-compactly generated. We leave the verification that this map is bijective as an exercise. $\square$

4.13. **Remark** We note that for each κ-condensed set X, the topological space $X(*)_{\mathrm{top}}$ is κ-compactly generated (essentially by definition of the topology). Moreover, a similar argument as in Proposition 4.12 shows that the construction sending $Y \in \mathrm{Top}$ to $\underline{Y}(*)_{\mathrm{top}}$ is also the left adjoint of the inclusion

$$\mathrm{Top}^{\kappa-\mathrm{cg}} \subseteq \mathrm{Top}.$$

Next, we will discuss how the notion of κ-condensed sets various when we change the strong limit cardinal κ in the definition.

4.14. **Theorem** *Let $\kappa' \geq \kappa$ be strong limit cardinals. The evident forgetful functor from* $\mathrm{Cond}_{\kappa'}(\mathrm{Set}) \to \mathrm{Cond}_{\kappa}(\mathrm{Set})$ *admits a left adjoint* $\mathrm{Cond}_{\kappa}(\mathrm{Set}) \to \mathrm{Cond}_{\kappa'}(\mathrm{Set})$*. This left adjoint is fully faithful, commutes with all colimits and with λ-small limits where $\lambda = \mathrm{cof}(\kappa)$.*

Proof. It is general non-sense that the claimed left adjoint exists: It is given by the composite

$$\mathrm{Cond}_{\kappa}(\mathrm{Set}) \subseteq \mathrm{PSh}(\mathrm{tdCH}_{\kappa}) \xrightarrow{i_!} \mathrm{PSh}(\mathrm{tdCH}_{\kappa'}) \longrightarrow \mathrm{Sh}(\mathrm{tdCH}_{\kappa'})$$

where the last functor is the sheafification functor and the middle functor is given by left Kan extension. Concretely, this says that the left adjoint sends a κ-condensed set X to the sheafification of the presheaf on $\mathrm{tdCH}_{\kappa'}$ given by

$$T \mapsto \operatorname*{colim}_{T \to S} X(S)$$

where the colimit runs over all maps $T \to S$ with $S \in \mathrm{tdCH}_{\kappa} \subseteq \mathrm{tdCH}_{\kappa'}$. To see the claim that it is fully faithful, we use the equivalence

$$\mathrm{Cond}_{\kappa}(\mathrm{Set}) \simeq \mathrm{Fun}^{\times}(\mathrm{edCH}_{\kappa}^{\mathrm{op}}, \mathrm{Set})$$

and likewise for κ'. We consider the diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\times}(\mathrm{edCH}_{\kappa}^{\mathrm{op}}, \mathrm{Set}) \arrow[r, dashed] \arrow[d] & \mathrm{Fun}^{\times}(\mathrm{edCH}_{\kappa'}^{\mathrm{op}}, \mathrm{Set}) \arrow[d] \\
\mathrm{Fun}(\mathrm{edCH}_{\kappa}^{\mathrm{op}}, \mathrm{Set}) \arrow[r] & \mathrm{Fun}(\mathrm{edCH}_{\kappa'}^{\mathrm{op}}, \mathrm{Set})
\end{tikzcd}$$

where the lower horizontal is the left Kan extension along the functor $\mathrm{edCH}_{\kappa}^{\mathrm{op}} \subseteq \mathrm{edCH}_{\kappa'}^{\mathrm{op}}$. We claim that this left Kan extension preserves functors which preserve finite products. To avoid confusion with taking opposite categories, let us generally show that if $\mathcal{C}_0 \to \mathcal{C}$ is a fully faithful functor which preserves finite products (and both categories admit finite products) and $\mathcal{D}$ is any category in which the product with a fixed objects preserves colimits (i.e. any closed category, like the category of sets), then the left Kan extension functor

$$\mathrm{Fun}(\mathcal{C}_0, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

preserves finite product preserving functors. For this, let $F\colon \mathcal{C}_0 \to \mathcal{D}$ be a product preserving functor. First, we have that $i_!(F)(*) = F(*) = *$ since the terminal object of $\mathcal{C}$ lies in $\mathcal{C}_0$. Second, we have

$$i_!(F)(c \times c') = \operatorname*{colim}_{s \in (\mathcal{C}_0)_{/c\times c'}} F(s).$$

Now consider the pair of functors

$$(\mathcal{C}_0)_{/c\times c'} \rightleftarrows (\mathcal{C}_0)_{/c} \times (\mathcal{C}_0)_{/c'}$$

where the functor from left to right sends $(s \to c \times c')$ to the pair $(s \to c, s \to c')$ and the functor from right to left sends a pair $(s \to c, s' \to c')$ to the morphism $(s \times s' \to c \times c')$. One checks that this is an adjoint pair, with the functor from right to left the right adjoint. Since right adjoints are cofinal, we deduce that

$$i_!(F)(c \times c') = \operatorname*{colim}_{(s \to c, s' \to c')} F(s \times s') = \operatorname*{colim}_{(s \to c, s' \to c')} F(s) \times F(s').$$

Then the claim follows from the fact that the functor $F(s) \times - \colon \mathcal{D} \to \mathcal{D}$ preserves colimits. Applying this to the functor $\mathrm{edCH}_\kappa^{\mathrm{op}} \to \mathrm{edCH}_{\kappa'}^{\mathrm{op}}$ in place of $\mathcal{C}_0 \to \mathcal{C}$ and Set in place of $\mathcal{D}$, we then deduce that the solid arrows in the above diagram restrict to the dashed arrow and that this dashed arrow is again the left adjoint of the forgetful functor. Since all the solid functors in the diagram are fully faithful, so is the dashed arrow. Now, since left adjoints commute with colimits, it finally remains to show that the left adjoint commutes with λ-small limits. For this, we wish to show that the relevant slice categories are λ-filtered, since in the category of sets, λ-filtered colimits commute with λ-small limits (this is a generalisation of the already used fact that (ω)-filtered colimits commute with finite limits), see Remark 4.15. The slice category under consideration is the category of maps $(\mathrm{edCH}_\kappa^{\mathrm{op}})_{/T}$, i.e. which consists objects given by maps $T \to S$ with $S \in \mathrm{edCH}_\kappa$ and morphisms given by commutative triangles. To show that this category is λ-filtered, we have to show that for any diagram S_i indexed over a λ-small category I, with compatible maps $T \to S_i$, there exists a κ-small extremally disconnected space S with maps $T \to S \to S_i$ in a compatible way. In first iteration, we may consider $\lim_I S_i$ which is κ-small and possibly not extremally disconnected, but it comes with maps $T \to \lim_I S_i \to S_i$ as needed. We may then find a surjection $S \to \lim_I S_i$ with S extremally disconnected and also κ-small. Since T is extremally disconnected, the map $T \to \lim_I S_i$ lifts to a map $T \to S$, by an earlier exercise which characterises the extremally disconnected compact Hausdorff spaces as the projective ones. □

4.15. **Remark** In the above proof we have used the notion of a regular cardinal κ. There are the following equivalent characterisations of regularity of κ:

(1) one has $\mathrm{cof}(\kappa) = \kappa$, and
(2) the category Set_κ of κ-small sets is closed under κ-small colimits.

For any cardinal κ, we have that $\mathrm{cof}(\mathrm{cof}(\kappa)) = \mathrm{cof}(\kappa)$, i.e. $\mathrm{cof}(\kappa)$ is regular. Moreover, for a regular cardinal κ, one has that in the category of sets, κ-filtered colimits commute with κ-small limits. We could discuss this in an exercise.

4.16. **Remark** We remark that the proof of Theorem 4.14 relies heavily on the fact that κ-condensed sets are simply product preserving functors on some small category. In general, it will not at all be clear that such left Kan extensions according to enlarging κ are fully faithful (simply because in general the left Kan extension on the level of presheaves will not preserve sheaves).

4.17. **Remark** In this remark, we want to give an alternative proof of the fully faithfulness of $\mathrm{Cond}_\kappa(\mathrm{Set}) \to \mathrm{Cond}_{\kappa'}(\mathrm{Set})$ which is actually not so different, in hindsight. For this, we first consider the following general construction. Let $\mathcal{C}$ be an essentially small category which admits finite coproducts (the example to keep in mind is edCH_κ). We denote by $\mathrm{PSh}_\Sigma(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ the full subcategory generated by the Yoneda image under 1-sifted colimits. Concretely, $\mathrm{PSh}_\Sigma(\mathcal{C})$ is the smallest subcategory of $\mathrm{PSh}(\mathcal{C})$ which contains the Yoneda image and is closed under 1-sifted colimits (that is colimits indexed over a category I such that the diagonal functor $I \to I \times I$ is 1-cofinal). Equivalently, it is the smallest subcategory of $\mathrm{PSh}(\mathcal{C})$ containing the Yoneda image and is closed under filtered colimits and coequalisers, though this statement requires a proof.

We now note that there is a canonical inclusion

$$\mathrm{PSh}_\Sigma(\mathcal{C}) \subseteq \mathrm{PSh}^\times(\mathcal{C}) = \mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}).$$

Indeed, the for any object $c \in \mathcal{C}$, $y(c)$ preserves products because $\mathrm{Hom}_\mathcal{C}(-, c)$ sends coproducts in $\mathcal{C}$ to products of sets. Furthermore, the category of product preserving functors is closed under 1-sifted colimits: Let $I \to \mathrm{PSh}^\times(\mathcal{C})$ be a functor with $I \to I \times I$ cofinal. Then

$$\begin{aligned}
\operatorname*{colim}_{i\in I} F_i(c \amalg c') &= \operatorname*{colim}_{i\in I} F_i(c) \times F_i(c') \\
&\xrightarrow{\Delta} \operatorname*{colim}_{(i,j)\in I\times I} F_i(c) \times F_j(c') \\
&= \operatorname*{colim}_{i\in I} F_i(c) \times \operatorname*{colim}_{j\in I} F_j(c')
\end{aligned}$$

and the map labelled Δ, which is induced by the diagonal of I, is an isomorphism since I is 1-sifted.

We now show that the just observed inclusion is in fact an equality. For this, let $X \in \mathrm{PSh}^\times(\mathcal{C})$. It will suffice to show that the canonical way of writing X as a colimit of representables is indexed over a 1-sifted diagram. In other words, we need to show that $\mathcal{C}_{/X}$ is 1-sifted, provided X preserves products. This follows from the simple observation that the diagonal functor $\mathcal{C}_{/X} \to \mathcal{C}_{/X} \times \mathcal{C}_{/X}$ admits a left adjoint given by

$$\mathcal{C}_{/X} \times \mathcal{C}_{/X} \to \mathcal{C}_{/X} \quad (c \to X, c' \to X) \mapsto (c \amalg c' \to X)$$

where we note that maps $c \amalg c' \to X$, by the Yoneda lemma, are given by $X(c \amalg c') = X(c) \times X(c')$. We deduce the equality

$$\mathrm{PSh}_\Sigma(\mathcal{C}) = \mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$$

and note that the argument used in the existence of the left adjoint shows that the Yoneda embedding $\mathcal{C} \to \mathrm{PSh}_\Sigma(\mathcal{C})$ preserves finite coproducts. Moreover, the image of the Yoneda embedding consists of *compact projective* objects, that is of objects T such that $\mathrm{Hom}(T, -)$ preserves 1-sifted colimits. Indeed, this simply follows from the fact that 1-sifted colimits in $\mathrm{PSh}_\Sigma(\mathcal{C})$ are calculated in $\mathrm{PSh}(\mathcal{C})$, since the (1-sifted) colimit of product preserving presheaves is again product preserving.

The construction $\mathcal{C} \mapsto \mathrm{PSh}_\Sigma(\mathcal{C})$ has a universal property, similar to the universal property of all presheaves: Any functor $f \colon \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is a category with 1-sifted colimit extends essentially uniquely to a functor $\bar{f} \colon \mathrm{PSh}_\Sigma(\mathcal{C}) \to \mathcal{D}$ which preserves 1-sifted colimits. In addition:

(1) if $\mathcal{D}$ admits coproducts and f preserves coproducts, then $\bar{f}$ preserves all colimits (in fact, is a left adjoint).

(2) if f is fully faithful and the image of f consists of compact projective objects, then the functor $\bar{f}$ is fully faithful.

In particular, this applies to the the following situation. Suppose we are given a fully faithful functor $f : \mathcal{C} \to \mathcal{C}'$ which preserves finite coproducts between categories which admit all finite coproducts. Then the composite $\mathcal{C} \to \mathcal{C}' \to \mathrm{PSh}_\Sigma(\mathcal{C}')$ is again fully faithful, preserves coproducts and has image given by compact projectives. Therefore, the resulting functor

$$\mathrm{PSh}_\Sigma(\mathcal{C}) \to \mathrm{PSh}_\Sigma(\mathcal{C}')$$

preserves all colimits and is again fully faithful. This applies to the fully faithful inclusion $\mathrm{edCH}_\kappa \to \mathrm{edCH}_{\kappa'}$ and therefore gives that the functor

$$\mathrm{Cond}_\kappa(\mathrm{Set}) \to \mathrm{Cond}_{\kappa'}(\mathrm{Set})$$

preserves colimits and is fully faithful.

Exercise. Prove the above mentioned universal property of $\mathcal{C} \mapsto \mathrm{PSh}_\Sigma(\mathcal{C})$.

4.18. **Definition** We now define $\mathrm{Cond}(\mathrm{Set}) = \bigcup_\kappa \mathrm{Cond}_\kappa(\mathrm{Set})$.

4.19. **Remark** The same argument as in Theorem 4.14 also shows that for $\kappa' \geq \kappa$, the left adjoint functor

$$\mathrm{Cond}_\kappa(\mathcal{D}) \to \mathrm{Cond}_{\kappa'}(\mathcal{D})$$

is fully faithful, and preserves colimits and κ-small limits, at least if $\mathcal{D}$ is bicomplete and κ-filtered colimits in $\mathcal{D}$ commute with κ-small limits and we take $\mathrm{Fun}^\times(\mathrm{edCH}_\kappa, \mathcal{D})$ as definition of $\mathrm{Cond}_\kappa(\mathcal{D})$ - or we add the assumptions on $\mathcal{D}$ which assure that the a priori different notions of $\mathrm{Cond}_\kappa(\mathcal{D})$ agree, see Remark 4.7.

4.20. **Definition** We let $\mathrm{Cond}(\mathrm{Ab}) = \bigcup_\kappa \mathrm{Cond}_\kappa(\mathrm{Ab})$.

Exercise. Show that the functors $\mathrm{Cond}_\kappa(\mathrm{Set}) \to \mathrm{Cond}(\mathrm{Set})$ and $\mathrm{Cond}_\kappa(\mathrm{Ab}) \to \mathrm{Cond}(\mathrm{Ab})$ preserve colimits and $\mathrm{cof}(\kappa)$-small limits.

4.21. **Remark** By construction, a condensed X set is a κ-condensed set for some κ. We may then form the left Kan extension of X along the inclusion $\mathrm{edCH}_\kappa^{\mathrm{op}} \to \mathrm{edCH}^{\mathrm{op}}$ and obtain a functor $\bar{X} : \mathrm{edCH}^{\mathrm{op}} \to \mathrm{Set}$. Since the transition maps in the definition of $\mathrm{Cond}(\mathrm{Set})$ are given by left Kan extension along $\mathrm{edCH}_\kappa^{\mathrm{op}} \to \mathrm{edCH}_{\kappa'}^{\mathrm{op}}$, the resulting functor $\bar{X} : \mathrm{edCH}^{\mathrm{op}} \to \mathrm{Set}$ does not depend on the choice of κ. We now observe that $\bar{X}$ preserves coproducts, simply because any two objects of edCH belong to edCH_τ for a suitable τ. Choosing the κ for X larger than τ, the claim follows since κ-condensed sets are product preserving functors on $\mathrm{edCH}_\kappa^{\mathrm{op}}$.

In Proposition 4.32 below, we characterise which product preserving functors $\mathrm{edCH}^{\mathrm{op}} \to \mathrm{Set}$ are condensed sets, without saying the words "is left Kan extended from $\mathrm{edCH}_\kappa^{\mathrm{op}}$".

In what follows we will need a more direct criterion saying which functors on $\mathrm{edCH}^{\mathrm{op}}$ are left Kan extended from their restriction to $\mathrm{edCH}_\kappa^{\mathrm{op}}$ for some strong limit cardinal κ, or more concretely which product preserving functors $\mathrm{edCH}^{\mathrm{op}} \to \mathrm{Set}$ are condensed sets. The following is essentially taken from [Man22]. First, we have to quickly introduce a variant of κ-condensed sets for κ a regular (rather than strong limit) cardinal.

We will need the following observation.

4.22. **Remark** We consider the category Pro(FinSet) which is by definition the full subcategory of $\mathrm{Fun}(\mathrm{FinSet}, \mathrm{Set})^{\mathrm{op}}$ generated under small cofiltered limits by the Yoneda image $\mathrm{FinSet} \to \mathrm{Fun}(\mathrm{FinSet}, \mathrm{Set})^{\mathrm{op}}$. There is a canonical functor $\mathrm{Pro}(\mathrm{FinSet}) \to \mathrm{tdCH}$ sending a formal cofiltered diagram $(S_i)_{i\in I}$ of finite sets to $S = \lim_I S_i$, the limit taken in topological spaces. We claim that this functor is an equivalence of categories. We have already argued that it is essentially surjective, see Theorem 2.48, so it remains to show that it is fully faithful. For this, let $(T_j)_{j\in J}$ be another formal cofiltered diagram of finite sets. We have

$$\mathrm{Hom}_{\mathrm{Pro(FinSet)}}((S_i)_{i\in I}, (T_j)_{j\in J}) = \lim_{j\in J} \operatorname*{colim}_{i\in I} \mathrm{Hom}_{\mathrm{FinSet}}(S_i, T_j)$$

as follows from the definition of Pro(FinSet). Furthermore, we have

$$\mathrm{Hom}_{\mathrm{Top}}(\lim_{i\in I} S_i, \lim_{j\in J} T_j) = \lim_{j\in J} \mathrm{Hom}_{\mathrm{Top}}(\lim_{i\in I} S_i, T_j)$$

so it suffices to prove the fully faithfulness for $(T_j)_{j\in J}$ replaced by a single finite set T. In other words, we need to show that the canonical map

$$\operatorname*{colim}_{i\in I} \mathrm{Hom}_{\mathrm{Top}}(S_i, T) \to \mathrm{Hom}_{\mathrm{Top}}(\lim_{i\in I} S_i, T)$$

is a bijection. We leave this as an exercise.

Exercise. Show that the canonical map

$$\operatorname*{colim}_{i\in I} \mathrm{Hom}_{\mathrm{Top}}(S_i, T) \to \mathrm{Hom}_{\mathrm{Top}}(\lim_{i\in I} S_i, T)$$

is a bijection.

4.23. **Remark** Of course, dual to the category of pro objects is the category of ind objects, $\mathrm{Ind}(\mathcal{C})$, which is the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) = \mathrm{PSh}(\mathcal{C})$ generated under filtered colimits from the Yoneda image $\mathcal{C} \subseteq \mathrm{PSh}(\mathcal{C})$

4.24. **Definition** Let κ be a cardinal. An object $X \in \mathrm{tdCH}$ is called κ-cocompact if for each κ-cofiltered diagram $S\colon I \to \mathrm{tdCH}$, the canonical map

$$\operatorname*{colim}_{i\in I} \mathrm{Hom}(S_i, X) \to \mathrm{Hom}(\lim_{i\in I} S_i, X)$$

is an isomorphism. We let tdCH^{κ} be the full subcategory on the κ-cocompact objects.

4.25. **Lemma** *Let $X \in \mathrm{tdCH}$ and κ be an uncountable regular cardinal. Then X is κ-cocompact if and only if it can be written as a κ-small and ω-cofiltered limit of finite (discrete) spaces.*

Proof. Let us also note that since κ is regular, κ-small cofiltered limits of κ-cocompact objects are κ-cocompact and that finite discrete spaces are κ-cocompact as we have essentially also argued in Remark 4.22. The converse is a general characterisation of κ-compact objects in compactly generated categories as the ones that can be written as κ-small filtered colimits of representables. Applying this to $\mathrm{tdCH}^{\mathrm{op}} = \mathrm{Pro}(\mathrm{FinSet})^{\mathrm{op}} = \mathrm{Ind}((\mathrm{FinSet})^{\mathrm{op}})$ gives the claim. This makes use of the fact that finite sets have finite limits, so $\mathrm{FinSet}^{\mathrm{op}}$ has finite colimits and hence its ind-category is compactly generated. ☐

4.26. **Corollary** *Let τ be a regular cardinal. Then there is a strong limit cardinal $\kappa \geq \tau$ such that* $\mathrm{tdCH}^\tau \subseteq \mathrm{tdCH}_\kappa$*, i.e. such that all τ-cocompact profinite spaces are κ-small.*

Proof. Choose a strong limit cardinal κ such that $\mathrm{cof}(\kappa) \geq \tau$. The claim then follows from Lemma 4.25 and the fact that $\mathrm{cof}(\kappa) \geq \tau$ implies that κ-small sets are closed under τ-small limits. □

Conversely, we have the following.

4.27. **Corollary** *Let τ be a cardinal. Then there exists a regular cardinal $\kappa \geq \tau$ such that* $\mathrm{tdCH}_\tau \subseteq \mathrm{tdCH}^\kappa$*, i.e. such that all τ-small profinite spaces are κ-cocompact.*

Proof. The canonical way of writing a profinite space as a limit of finite discrete spaces, see Theorem 2.48, has the property that the indexing diagram is ω-filtered. Furthermore, it consists of disjoint union decompositions of X and is hence itself τ'-small for τ' suitably big and independent of X (it only depends on τ). Therefore, by Lemma 4.25, it suffices to find a regular cardinal $\kappa \geq \tau'$. □

We note that the covering families of type (1) and of type (2) or (3) again form a quasi-covering on the category tdCH^κ. The topology induced by this quasi covering is again called the condensed topology.

4.28. **Definition** Let κ be a regular cardinal. We define $\mathrm{Cond}_\kappa(\mathrm{Set}) = \mathrm{Sh}(\mathrm{tdCH}^\kappa)$ to be sheaves on κ-cocompact profinite spaces with respect to the condensed topology.

4.29. **Proposition** *Let $\kappa' \geq \kappa$ be uncountable regular cardinals. Then the restriction functor* $\mathrm{Cond}_{\kappa'}(\mathrm{Set}) \to \mathrm{Cond}_\kappa(\mathrm{Set})$ *has a left adjoint which is fully faithful. Moreover, an object* $X \in \mathrm{Cond}_{\kappa'}(\mathrm{Set})$ *lies in the image of the left adjoint if and only if for each κ-cofiltered diagram $S\colon I \to \mathrm{tdCH}^{\kappa'}$ whose limit is again κ'-cocompact, the canonical map*

$$\operatorname*{colim}_{i\in I} X(S_i) \to X(S)$$

is an isomorphism.

Proof. The claim is that the composite

$$\mathrm{Cond}_\kappa(\mathrm{Set}) \subseteq \mathrm{PSh}(\mathrm{tdCH}^\kappa) \xrightarrow{i_!} \mathrm{PSh}(\mathrm{tdCH}^{\kappa'})$$

has image contained in sheaves, and hence the functor is fully faithful as needed. As in the proof of Theorem 4.14, for $X \in \mathrm{Cond}_\kappa(\mathrm{Set})$, we have that $i_!(X)$ preserves finite products and hence satisfies descent for families of type (2). It hence suffices to show that $i_!(X)$ also satisfies descent for families of the form $\{T' \to T\}$ consisting of a single surjection between κ'-cocompact objects of tdCH. We claim that one can the write this map as a κ-cofiltered limit of surjections $\{T_i' \to T_i\}_{i\in I}$ between κ-cocompact objects, see Lemma 4.30 below.

We claim that the resulting functor $I \to [(\mathrm{tdCH}^\kappa)^{\mathrm{op}}]_{/T}$, sending i to the map $T \to T_i$ is cofinal: To see this, we need to argue that for each object $S \in [(\mathrm{tdCH}^\kappa)^{\mathrm{op}}]_{/T}$, i.e. for each map $T \to S$ with S κ-cocompact, the slice $I_{S/}$ is connected. Objects in this slice consist of pairs $(i, T_i \to S)$ such that the composite $T \to T_i \to S$ is the given map. We then see that this slice is non-empty since S is κ-cocompact, so that any map $T \to S$ factors through a map $T_i \to S$ for some $i \in I$. To see that the slice is also connected, consider two such elements, say $(i, T_i \to S)$ and $(j, T_j \to S)$. Then both these maps determine the same element of $\mathrm{Hom}(T, S) = \mathrm{colim}_k \mathrm{Hom}(T_k, S)$ and since this colimit is κ-filtered, we see that they must already agree for some $k \in I$. This means that for this k, we have that the two maps $T_k \to S$ - the one factoring through T_i and the other factoring through T_j - agree, showing connectivity of the slice. Having all this, we consider the diagram

$$i_!(X)(T) \to i_!(X)(T') \rightrightarrows i_!(X)(T' \times_T T').$$

Using the above cofinality (three times) we see that this diagram is isomorphic to the diagram

$$\operatorname*{colim}_{i \in I} X(T_i) \to \operatorname*{colim}_{i \in I} X(T_i') \rightrightarrows \operatorname*{colim}_{i \in I} X(T_i' \times_{T_i} T_i')$$

which is a κ-filtered colimit of equaliser diagrams since $X \in \mathrm{Cond}_\kappa(\mathrm{Set})$. Since κ-filtered colimits commute with κ-small limits (and in particular finite limits like equalisers), we deduce that $i_!(X) \in \mathrm{Cond}_{\kappa'}(\mathrm{Set})$.

To see the second part, let $S_{(-)} \colon I \to \mathrm{tdCH}^{\kappa'}$ be a κ-cofiltered diagram such that its limit S is again κ'-cocompact. First, assume that X lies in the image of the left adjoint, i.e. is given by $i_!(Y)$ for $Y \in \mathrm{Cond}_\kappa(\mathrm{Set})$ on the level of presheaves. We have

$$i_!(Y)(S) = \operatorname*{colim}_{T \in [(\mathrm{tdCH}^\kappa)^{\mathrm{op}}]_{/S}} Y(T).$$

We claim that the canonical functor

$$\operatorname*{colim}_{i \in I} [(\mathrm{tdCH}^\kappa)^{\mathrm{op}}]_{/S_i} \to [(\mathrm{tdCH}^\kappa)^{\mathrm{op}}]_{/S}$$

is an equivalence of categories because S is the κ-cofiltered limit of the S_i's. Again, in more detail, we first show that the functor is essentially surjective. To do this, let $S \to T$ be an object of $[(\mathrm{tdCH}^\kappa)^{\mathrm{op}}]_{/S}$. Since T is κ-cocompact, and I is κ-cofiltered, we see that the map $S \to T$ factors as $S \to S_i \to T$ for some $i \in I$, showing essential surjectivity. Next, we show that the functor is fully faithful. For this, consider (without loss of generality) two objects in $[(\mathrm{tdCH}^\kappa)^{\mathrm{op}}]_{/S_i}$ for some $i \in I$, and view them as representing elements in the colimit. They are given by $\alpha \colon S_i \to T$ and $\alpha' \colon S_k \to T'$, say. Then we have that the Hom set in the colimit category is given by the set of maps $f \colon T \to T'$ such that the α' and $f\alpha$ represent the same map in $\mathrm{colim}_{i \in I} \mathrm{Hom}(S_i, T')$. We use again that T' is κ-cocompact to see that this colimit is simply given by $\mathrm{Hom}(S, T')$, showing that the Hom set in the colimit category is bijective to the set of all maps $T \to T'$ such that the induced triangle with the two maps from S agree, which is precisely the Hom set in $[(\mathrm{tdCH}^\kappa)^{\mathrm{op}}]_{/S}$ as claimed.

Therefore, we have

$$i_!(Y)(S) = \operatorname*{colim}_{T \in [(\mathrm{tdCH}^\kappa)^{\mathrm{op}}]_{/S}} Y(T) = \operatorname*{colim}_{i \in I} \operatorname*{colim}_{T_i \in [(\mathrm{tdCH}^\kappa)^{\mathrm{op}}]_{/S_i}} Y(T_i) = \operatorname*{colim}_{i \in I} i_!(Y)(S_i)$$

as needed. Conversely, if X satisfies the prescribed colimit property, then for $S \in \mathrm{tdCH}^{\kappa'}$, we have

$$X(S) = \operatorname*{colim}_{T \in [(\mathrm{tdCH}^\kappa)^{\mathrm{op}}]_{/S}} X(T)$$

since the colimit appearing here is κ-filtered and κ-cocompact objects are also κ'-cocompact and in addition we have that

$$S = \lim_{T \in [(\mathrm{tdCH}^\kappa)^{\mathrm{op}}]_{/S}} T.$$

In particular, we find that X agrees with the left Kan extension along its restriction to κ-cocompact objects as needed. $\square$

We owe the interested reader the following lemmata, for which we still not provide all details. For more of these, see [Man22, 2.1.5 % 2.1.6].

4.30. **Lemma** *Let $T' \to T$ be a surjection in* tdCH*. Then there exists a κ-cofiltered diagram $I \to \mathrm{Fun}([1], \mathrm{tdCH}^\kappa)$, sending i to a surjection $T'_i \to T_i$ such that* $\lim_i(T'_i \to T_i) = (T' \to T)$.

Proof. First note that every object of tdCH can be written as a κ-cofiltered limit of κ-cocompact objects. This is equivalent to the statement that every object of $\mathrm{tdCH}^{\mathrm{op}}$ can be written as a κ-filtered colimit of κ-compact objects. Now we note as earlier that $\mathrm{tdCH}^{\mathrm{op}} = \mathrm{Ind}(\mathrm{FinSet}^{\mathrm{op}})$ is a compactly generated presentable category. Such categories are κ-accessible for any regular cardinal. In particular, every object is indeed a κ-filtered colimit of κ-compact ones.

Now consider the category $J = \mathrm{Fun}([1], \mathrm{tdCH}^\kappa)_{(S \to T)/}$. Objects consist of commutative diagrams

$$\begin{tikzcd} S \arrow[r] \arrow[d] & T \arrow[d] \\ S_j \arrow[r] & T_j \end{tikzcd}$$

with S_j and T_j κ-cocompact. Then J is κ-cofiltered. There is in addition a canonical functor

$$J = \mathrm{Fun}([1], \mathrm{tdCH}^\kappa)_{(S \to T)/} \to \mathrm{tdCH}^\kappa_{T/} = J_0$$

sending a commutative diagram as above to the morphism $T \to T_j$. This functor is essentially surjective (pick $S_j = T_j$). Writing $T = \lim_i T_i$ for some κ-cofiltered index category. As in the proof of Proposition 4.29 the resulting functor $I \to \mathrm{tdCH}^\kappa_{T/}$ is cofinal. Then we may consider the category $J \times_{J_0} I$, claim that it is again κ-cofiltered, and now comes with a surjective functor to I. In addition, the functor $J \times_{J_0} I \to J$ is also cofinal. Consider $I' \subseteq J \times_{J_0} I$ be the subcategory where the map $S_j \to T_j$ is surjective. We claim that this inclusion is also cofinal, since one can replace the T_j be the image of the map $S_j \to T_j$ - note here that the map $T \to T_j$ factors through this image since $S \to T$ is assumed to be surjective. We now need to use the fact that this image, being a closed subset of a κ-cocompact object is again κ-cocompact, see Lemma 4.31. □

4.31. **Lemma** *Let $T \in \mathrm{tdCH}^\kappa$ and $S \subseteq T$ a closed subset. Then $S \in \mathrm{tdCH}^\kappa$.*

Proof. By Lemma 4.25 we know that we can write $T = \lim_i T_i$ with I κ-small filtered and all T_i finite discrete spaces. Let S_i be the image of the map $S \to T \to T_i$. We claim that $S = \lim_i S_i$, showing that S is κ-cocompact as desired. □

4.32. **Proposition** *Let $X \colon \mathrm{edCH}^{\mathrm{op}} \to \mathrm{Set}$ be a product preserving functor. Then $X \in \mathrm{Cond}(\mathrm{Set})$ if and only if there exists a regular cardinal κ such that for all κ-cofiltered diagrams $T \colon I \to \mathrm{edCH}$ and $T = \lim_i T_i$, we have that the canonical map*

$$\operatorname*{colim}_{i \in I} X(T_i) \longrightarrow X(T)$$

is an isomorphism. If this is so, then for $T = \lim_i T_i$ a κ-cofiltered limit in tdCH*, the same map is an isomorphism as well.*

Proof. Assume the condition on X holds. Pick a strong limit cardinal τ such that $\mathrm{cof}(\tau) \geq \kappa$. We may view X as an object of $\mathrm{Cond}_\tau(\mathrm{Set})$ and claim that the resulting functor $\bar{X}$ of Remark 4.21 is canonically isomorphic to X. To see this, let $T \in \mathrm{edCH}$. Then

$$\bar{X}(T) = \operatorname*{colim}_{S \in (\mathrm{edCH}^{\mathrm{op}}_\tau)_{/T}} X(S) \to X(T)$$

is the canonical map we wish to show is an isomorphism. We have argued in Theorem 4.14 that this colimit is $\mathrm{cof}(\tau)$-filtered and hence also κ-filtered. Therefore, the map is an isomorphism as needed.

Conversely, let $X \in \mathrm{Cond}(\mathrm{Set})$ and. By definition, we can find a strong limit cardinal τ such that $X \in \mathrm{Cond}_\tau(\mathrm{Set})$. Pick a regular cardinal $\kappa \geq \tau$ such that $\mathrm{tdCH}_\tau \subseteq \mathrm{tdCH}^\kappa$. We claim this is the κ we are looking for. To show this, let $T = \lim_i T_i$ be a κ-cofiltered diagram. Pick $\kappa' \geq \kappa$ regular so that T and all T_i are κ'-cocompact. Finally, pick $\tau' \geq \kappa'$ a strong limit cardinal. Then we have the following functors

$$\mathrm{Cond}_\tau(\mathrm{Set}) \to \mathrm{Cond}_\kappa(\mathrm{Set}) \to \mathrm{Cond}_{\kappa'}(\mathrm{Set}) \to \mathrm{Cond}_{\tau'}(\mathrm{Set}) \to \mathrm{Cond}(\mathrm{Set}).$$

The first three functors are induced by left Kan extension along the inclusions

$$\mathrm{tdCH}_\tau \subseteq \mathrm{tdCH}^\kappa \subseteq \mathrm{tdCH}^{\kappa'} \subseteq \mathrm{tdCH}_{\tau'}.$$

In particular, the image of X, viewed as an object of $\mathrm{Cond}_\tau(\mathrm{Set})$ is precisely the X we have started with. We may then apply Proposition 4.29 the the image of X in $\mathrm{Cond}_{\kappa'}(\mathrm{Set})$ and the diagram $(T_i)_{i \in I}$. $\square$

4.33. **Lemma** *The category* Cond(Set) *is cocomplete and complete. Limits and filtered colimits are calculated pointwise.*

Proof. Consider a diagram $X\colon I \to \mathrm{Cond(Set)}$. One can choose κ such that $\lambda = \mathrm{cof}(\kappa)$ is at least as large as the cardinality of I and such that for all $i \in I$, $X_i \in \mathrm{Cond}_\kappa(\mathrm{Set})$. We have then argued that for $\kappa' \geq \kappa$, the left Kan extension $\mathrm{Cond}_\kappa(\mathrm{Set}) \to \mathrm{Cond}_{\kappa'}(\mathrm{Set})$ preserves the limit over I, and moreover, limits in sheaves are always computed in presheaves. This also shows that the functor $\mathrm{Cond}_\kappa(\mathrm{Set}) \to \mathrm{Cond(Set)}$ preserves the limit. The same argument holds true for colimits, where in the filtered case we again use that filtered colimits of sheaves are sheaves, so that filtered colimits in sheaves can be calculated pointwise. □

4.34. **Remark** As before, where we have introduced a functor $\mathrm{Top} \to \mathrm{Cond}_\kappa(\mathrm{Set})$ for each κ, we might wonder about a functor $\mathrm{Top} \to \mathrm{Cond(Set)}$. We note that we ought to be careful here: For each κ, the functor $\mathrm{Top} \to \mathrm{Cond}_\kappa(\mathrm{Set})$ is a right adjoint, and by construction commutes with the forgetful functors $\mathrm{Cond}_{\kappa'}(\mathrm{Set}) \to \mathrm{Cond}_\kappa(\mathrm{Set})$. There is no general reason why it would also commute with the left adjoints $\mathrm{Cond}_\kappa(\mathrm{Set}) \to \mathrm{Cond}_{\kappa'}(\mathrm{Set})$ used to form Cond(Set). Nevertheless, one may wonder whether the formula

$$\underline{(-)}\colon \mathrm{Top} \to \mathrm{Fun}^\times(\mathrm{edCH}^{\mathrm{op}}, \mathrm{Set}) \quad X \mapsto (T \mapsto \mathrm{Hom}_{\mathrm{Top}}(T, X))$$

gives rise to a condensed set. The answer is, however, *no* in general.

To see this, consider S the Sierpinski space, i.e. a space with 2 point precisely one of which is open. Then the functor $\underline{S}$ sends an extremally disconnected space to its set of open subsets. This functor is not a condensed set, as we work out in the exercise below.

Exercise. Show that the functor sending an extremally disconnected space T to its set of open (or equivalently closed) subsets is not a condensed set.

The above remark suggests that some separation hypothesis might help in constructing condensed sets from topological spaces. This is indeed the case. Before we come to the details, we first introduce some terminology for condensed sets.

4.35. **Definition** A map $f\colon X \to Y$ of condensed sets is called

(1) injective, if for all $T \in \mathrm{tdCH}$, the map $X(T) \to Y(T)$ is injective,
(2) surjective, if for all $y \in Y(T)$, there exists a cover $\mathcal{T} = \{T_i \to T\}_{i \in I}$ of T such that each $y_i = p_i^*(y) \in Y(T_i)$ lifts to an element $x_i \in X(T_i)$.

4.36. **Remark** Concretely, the above simply means that if $f\colon X \to Y$ is a map in $\mathrm{Cond}_\kappa(\mathrm{Set})$ and $f_{\kappa'}$ is the resulting map in $\mathrm{Cond}_{\kappa'}(\mathrm{Set})$ for any $\kappa' \geq \kappa$, then $f_{\kappa'}$ is injective/surjective in the usual sense of sheaves.

Exercise. A map of condensed sets is an isomorphism if and only if it is injective and surjective.

4.37. **Definition** A condensed set X is called

(1) quasicompact (qc for short) if for any collection $\{X_i \to X\}_{i \in I}$ such that $\amalg_I X_i \to X$ is surjective, there exists a finite subset $J \subseteq I$ such that $\amalg_J X_j \to X$ is also surjective.
(2) quasiseparated (or Hausdorff, and qs for short) if for any $T, T' \in \mathrm{tdCH}$ and any two maps $T \to X$, and $T' \to X$, the pullback $T \times_X T'$ is a quasicompact condensed set.

(3) $T1$ if for any $T \in \mathrm{tdCH}$ and maps $T \to X$ and $* \to X$, the pullback $* \times_X T$ is quasicompact.

Exercise. Show that $T1$ condensed sets are closed under limits in Cond(Set).

In what follows we let HausTop denote the full subcategory of Top on Hausdorff topological spaces and $T1$Top the full subcategory on $T1$-spaces (i.e. where points are closed).

4.38. **Theorem** *There is a canonical functor* $(\underline{-})\colon T1\mathrm{Top} \to T1\mathrm{Cond(Set)}$ *given by* $\underline{X}(T) = \mathrm{Hom}_{\mathrm{Top}}(T, X)$. *This functor has a left adjoint* $X \mapsto X(*)_{\mathrm{top}}$, *where we view* $X \in \mathrm{Cond}_\kappa(\mathrm{Set})$ *for suitable* κ.

Proof. First, we show that $\underline{X}$ is a condensed set. We need to show that the association $T \mapsto \mathrm{Hom}_{\mathrm{Top}}(T, X)$, as a functor $\mathrm{edCH}^{\mathrm{op}} \to \mathrm{Set}$, is left Kan extended from $\mathrm{edCH}^{\mathrm{op}}_\kappa$ for some strong limit cardinal κ, or in other words that the canonical map

$$\operatorname*{colim}_{T' \in (\mathrm{edCH}^{\mathrm{op}}_\kappa)_{/T}} \mathrm{Hom}(T', X) \to \mathrm{Hom}(T, X) \tag{2}$$

is a bijection for all $T \in \mathrm{edCH}$. We claim that any strong limit cardinal κ with $|X| < \mathrm{cof}(\kappa)$ does the job. We first argue that the map is injective, this is true for X any topological space, and then use the $T1$-condition on X to obtain surjectivity.

To see injectivity, we consider $f_0\colon T_0 \to X$ and $f_1\colon T_1 \to X$ elements in the left hand side of (2) whose image in $\mathrm{Hom}(T, X)$ agree. First, we consider the diagram

$$\begin{tikzcd} & T' \arrow[d, two heads] \\ T \arrow[ur, dashed, "\alpha"] \arrow[r] & T_0 \times T_1 \arrow[r, shift left] \arrow[r, shift right] & X \end{tikzcd}$$

with $T' \in \mathrm{edCH}_\kappa$, we reduce to the case where $T_0 = T_1$ - note that the dashed arrow exists since T is a projective object in compact Hausdorff spaces. We then consider $T'' = \mathrm{Im}(\alpha) \subseteq T'$ which is a closed subspace since α is a continuous map between compact Hausdorff spaces and hence closed. Therefore $T'' \in \mathrm{tdCH}_\kappa$ and we may choose another surjection $\bar{T} \to T''$ with $\bar{T} \in \mathrm{edCH}_\kappa$. Using again the projectivity of T the map $T \to T''$ can be lifted to a map $T \to \bar{T}$. By construction, the two maps $\bar{T} \to X$ agree, so injectivity follows as wanted.

Now we argue that the map (2) is surjective and therefore fix a map $T \to X$. To do so, we first claim that it suffices to find an element $T' \in (\mathrm{edCH}^{\mathrm{op}}_\kappa)_{/T}$ such that the two canonical maps

$$T \times_{T'} T \rightrightarrows T \to X$$

agree. Indeed, if this is so, let us again denote by $\alpha\colon T \to T'$ the map and by $\bar{T}$ its image. Then the diagram

$$T \times_{T'} T \rightrightarrows T \to \bar{T}$$

is a coequaliser diagram (the map from the coequaliser to $\bar{T}$ is a continuous bijection between compact Hausdorff spaces). Therefore, we obtain a factorisation

$$T \to \bar{T} \to X$$

of the fixed map through $\bar{T}$. Again, $\bar{T}$ is a closed subset of T' and therefore we find $\bar{T} \in \mathrm{tdCH}_\kappa$. Then we may find a surjection $\tilde{T} \to \bar{T}$ with $\tilde{T} \in \mathrm{edCH}_\kappa$, and lift the map $T \to \bar{T}$ to $\tilde{T}$. This provides a factorisation

$$T \to \tilde{T} \to X$$

of the fixed map through an object of edCH_κ, which is what we need.

It now remains to show that there exists an element $T' \in (\mathrm{edCH}_\kappa^{\mathrm{op}})_{/T}$ such that the two canonical maps

$$T \times_{T'} T \rightrightarrows T \to X$$

agree. To do this, we fix $x \neq y$ in X and first claim that there is $T_{x,y} \in (\mathrm{edCH}_\kappa^{\mathrm{op}})_{/T}$ such that the map

$$T \times_{T_{x,y}} T \to T \times T \to X \times X$$

does not have the pair (x, y) in its image. Let us denote by $F_{T'}$ the map $T \times_{T'} T \to X \times X$. We note that

$$\lim_{T' \in (\mathrm{edCH}_\kappa^{\mathrm{op}})_{/T}} T \times_{T'} T = T$$

and similarly that

$$\lim_{T' \in (\mathrm{edCH}_\kappa^{\mathrm{op}})_{/T}} F_{T'}^{-1}(x, y) = \varnothing$$

since the two maps $T \times_T T \to X$ agree, of course. Here, we have used that the diagrams over which we take limits are connected (in fact $\mathrm{cof}(\kappa)$-filtered) so that the limit over a constant functor is simply the object itself. Now, since X is $T1$, we find that $F_{T'}^{-1}(x, y)$ is a closed subspace of $T \times_{T'} T$ which is compact Hausdorff, and hence also $F_{T'}^{-1}(x, y)$ is compact Hausdorff. We can then use the following exercise to conclude that there exists a T' such that $F_{T'}^{-1}(x, y) = \varnothing$.

We have now argued that for each pair of distinct points $x \neq y$ in X, we find $T_{x,y} \in (\mathrm{edCH}_\kappa^{\mathrm{op}})_{/T}$ such that $T \times T_{x,y} T \to X \times X$ does not have (x, y) in the image. Since $|X \times X| < \lambda$ and $(\mathrm{edCH}_\kappa^{\mathrm{op}})_{/T}$ is λ-filtered, we can find $T' \in (\mathrm{edCH}_\kappa^{\mathrm{op}})_{/T}$ such that every map $T \to T_{x,y}$ factors through the map $T \to T'$. In particular, the the map $T \times_{T'} T \to X \times X$ factors as

$$T \times_{T'} T \to T \times_{T_{x,y}} T \to X \times X$$

for every $x \neq y$, showing that the composite has image in the diagonal. This is simply saying that the two maps

$$T \times_{T'} T \rightrightarrows X$$

agree and therefore we conclude that $\underline{X}$ is a condensed set if X is a $T1$-space as needed.

We now show that $\underline{X}$ is a T1 condensed set. To do so, we consider a profinite set S and maps $* \to \underline{X} \leftarrow \underline{S}$ of condensed sets. These are equivalently given by maps of topological spaces $* \to X \leftarrow S$, since we have already argued that $\underline{X}$ and $\underline{S}$ are κ-condensed for an appropriate κ, and can then use Theorem 4.11. By Lemma 4.33, the pullback is calculated pointwise and therefore identifies with the condensed set given by the pullback $* \times_X S$ of topological spaces. This pullback is a closed subset of S and therefore again profinite. By an exercise below, we deduce that $\underline{X}$ is $T1$.

Now we show that the association $X \mapsto X(*)_{\mathrm{top}}$ is well-defined and a left adjoint to the above construction. First, we consider $X \in \mathrm{Cond}_\kappa(\mathrm{Set})$. Then we wish to argue that the topology on $X(*)_{\mathrm{top}}$ considered in Proposition 4.12 does not change when we view X as an object of $\mathrm{Cond}_{\kappa'}(\mathrm{Set})$ for $\kappa' \geq \kappa$. This follows from the fact that any map $S \to X$, with $S \in \mathrm{tdCH}_{\kappa'}$ factors through some $S' \in \mathrm{tdCH}_\kappa$, since X is κ-condensed, and the definition of the topology on $X(*)_{\mathrm{top}}$. Consequently, there is a functor $\mathrm{Cond}(\mathrm{Set}) \to \mathrm{Top}$ sending X to $X(*)_{\mathrm{top}}$. We note that, by construction, it takes values in compactly generated spaces. We now show that if $X \in T1\mathrm{Cond}_\kappa(\mathrm{Set})$, then $X(*)_{\mathrm{top}}$ is a $T1$-space. By definition of the topology on $X(*)_{\mathrm{top}}$, we need to show that for any $x \in X$ and for any map $S \to X$ with $S \in \mathrm{tdCH}_\kappa$, the pullback of topological spaces

$$\begin{tikzcd} P \arrow[r] \arrow[d] & S \arrow[d] \\ * \arrow[r, "x"] & X(*)_{\mathrm{top}} \end{tikzcd}$$

is a closed subspace of S, or equivalently, is a compact space. Let us consider the pullback

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & S \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

in κ-condensed sets. We have a continuous bijection $Q(*)_{\mathrm{top}} \to P$ (since the functor $Q \mapsto Q(*)$ preserves limits). By assumption, Q receives a surjection from a profinite set T, so we also have a surjection $T \to Q(*)_{\mathrm{top}} \to P$, showing that P is indeed compact.

Finally, let $A \in T1\mathrm{Top}$ and $X \in T1\mathrm{Cond}(\mathrm{Set})$. Choose κ such that X and $\underline{A}$ are in $\mathrm{Cond}_\kappa(\mathrm{Set})$. Then

$$\mathrm{Hom}_{T1\mathrm{Top}}(X(*)_{\mathrm{top}}, A) = \mathrm{Hom}_{\mathrm{Top}}(X(*)_{\mathrm{top}}, A) \cong \mathrm{Hom}_{\mathrm{Cond}_\kappa(\mathrm{Set})}(X, \underline{A}) = \mathrm{Hom}_{\mathrm{Cond}(\mathrm{Set})}(X, \underline{A}).$$

where the middle isomorphism holds by Proposition 4.12 and the outer equalities are clear since $T1\mathrm{Top} \subseteq \mathrm{Top}$ and $\mathrm{Cond}_\kappa(\mathrm{Set}) \subseteq \mathrm{Cond}(\mathrm{Set})$ are full subcategories. $\square$

Exercise. Let $T = \lim\limits_{i \in I}$ be an inverse limit of compact Hausdorff spaces. Show that if $T = \varnothing$, then there exists an $i \in I$ such that $T_i = \varnothing$.

Exercise. A condensed set X is quasicompact if and only if there exists a surjection $T \to X$ for $T \in \mathrm{tdCH}$. It is quasiseparated if and only if for $T \in \mathrm{tdCH}$ and maps $f, g \colon T \to X$, the pullback $T \times_X T$ is quasicompact.

In the following we shall make use of the following lemma.

4.39. **Lemma** *Let $X \in \mathrm{Cond(Set)}$ and let $T \in \mathrm{tdCH}$. Suppose that $X \subseteq \underline{T}$ is a sub condensed set, i.e. that there is an injective map $X \to \underline{T}$. If X is quasicompact, then $X = \underline{S}$ for some closed subset $S \subseteq T$.*

Proof. If X is an object of $\mathrm{qcCond}_\kappa(\mathrm{Set})$, we can find $A \in \mathrm{tdCH}_\kappa$ and a surjection $\underline{A} \to X$. We define S to be the image of the composite $A \to T$. Then we note that

$$\underline{A} \times_X \underline{A} = \underline{A} \times_{\underline{S}} \underline{A} = \underline{A \times_S A}$$

where the second equality holds because $\underline{\ }$ is a right adjoint and hence preserves limits and the first equality holds since $X(B) \to \underline{S}(B)$ is injective for all B. Now we note that

$$\underline{A} \times_X \underline{A} \rightrightarrows \underline{A} \to X$$

is a coequaliser diagram – this is a general fact in a topos and uses only that $A \to X$ is surjective. Now by the above we have

$$\underline{A} \times_X \underline{A} = \underline{A \times_S A}$$

so we find that X is isomorphic to the coequaliser of the two maps

$$\underline{A \times_S A} \rightrightarrows \underline{A}.$$

Now, similarly to the above, we also know that $S = \mathrm{Coeq}(A \times_S A \rightrightarrows A)$. Therefore it suffices to show that the canonical map

$$\mathrm{Coeq}(\underline{A \times_S A} \rightrightarrows \underline{A}) \to \underline{S}$$

is an isomorphism. To show this, simply map both sides to a general sheaf and use the sheaf condition for the type (3) family $\{A \to S\}$. $\square$

4.40. **Theorem** *The functor constructed in Theorem 4.38 induces an equivalence of categories* $\mathrm{CH} \simeq \mathrm{qcqsCond(Set)}$.

Proof. First, we argue that the functor $T1\mathrm{Top} \to \mathrm{Cond(Set)}$ obtained from Theorem 4.38 is fully faithful when restricted to compactly generated spaces. For this, we note that a compactly generated space is κ-compactly generated for some κ - the argument is similar as in the proof that a $T1$-space X is κ-condensed for some κ. Then we use Theorem 4.11. For $\mathrm{CH} \to \mathrm{Cond(Set)}$ the argument for fully faithfulness is of course even simpler, since any such is κ-small for some κ and therefore κ-compactly generated.

Next, we show that if $X \in \mathrm{CH}$, then $\underline{X} \in \mathrm{qcqsCond(Set)}$. Since any compact Hausdorff space admits a surjection from a profinite space, we find from an exercise above that $\underline{X} \in \mathrm{qcCond(Set)}$. Next we show that $\underline{X}$ is also quasiseparated. So let $T, T' \to X$ be maps of topological spaces (or equivalently, by what we have argued in the very beginning of this proof, maps of condensed sets). We need to show that the pullback $\underline{T} \times_{\underline{X}} \underline{T'}$ is a quasicompact condensed set. Since limits in $\mathrm{HausTop} \subseteq T1\mathrm{Top}$ are formed in Top, and the functor $(\underline{-})$ is a right adjoint, an exercise above implies that the pullback under consideration is given by $\underline{T \times_X T'}$. We have argued earlier that this pullback is a closed subset of $T \times T'$ since X is Hausdorff and therefore the pullback is compact as a topological space. We conclude again that $\underline{T \times_X T'}$ is a quasicompact condensed set.

It remains to show that the functor $\mathrm{CH} \to \mathrm{qcqsCond(Set)}$ is essentially surjective. So let $X \in \mathrm{qcqsCond(Set)}$. Choose a surjection $T \to X$ with $T \in \mathrm{tdCH}$. Form the pullback $T \times_X T$, which is quasicompact since X is quasiseparated. Since $T \times_X T \subseteq T \times T$, we apply

Lemma 4.39 and see that $T \times_X T = \underline{S}$ for some closed subset S of $T \times T$. Then by the argument the proof of Lemma 4.39, we find that

$$X = \mathrm{Coeq}(\underline{S} \rightrightarrows \underline{T} = \underline{\mathrm{Coeq}(S \rightrightarrows T)}$$

as needed. □

5. Condensed abelian groups

The next main goal is to study the category Cond(Ab) and show that it is as nice as one could possibly hope for. We recall that an object X of a category $\mathcal{C}$ is called compact projective if $\mathrm{Hom}_{\mathcal{C}}(X, -)$ preserves 1-sifted colimits. In what follows, a collection of objects $X_i \in \mathcal{C}$ is called a set of generators if the functor

$$\prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X_i, -) \colon \mathcal{C} \to \mathrm{Set}$$

is conservative.

5.1. **Theorem** *For each κ, the category* $\mathrm{Cond}_\kappa(\mathrm{Ab})$ *is an abelian category, generated by compact projective objects. In addition it satisfies the following properties:*

1. *it has all small limits and colimits,*
2. *products, coproducts, and filtered colimits are exact functors, and*
3. *for all sets I, filtered categories J_i for $i \in I$, and functors $M_i \colon J_i \to \mathrm{Cond}_\kappa(\mathrm{Ab})$, the canonical map*
$$\operatorname*{colim}_{j \in J_i} \prod_{i \in I} M_i(j) \to \prod_{i \in I} \operatorname*{colim}_{j \in J_i} M_i(j)$$
is an isomorphism.

The category Cond(Ab) *has the same properties.*

Proof. We identify $\mathrm{Cond}_\kappa(\mathrm{Ab})$ with the category of product preserving functors $\mathrm{edCH}_\kappa^{\mathrm{op}} \to \mathrm{Ab}$. Then we note that this category is closed (inside all functors) under arbitrary limits and colimits: This is because limits and colimits in abelian groups commute with products (since these are also coproducts). Now, the category Ab of abelian groups is abelian and satisfies the above three properties. It follows that any diagram category, in particular $\mathrm{Fun}(\mathrm{edCH}_\kappa^{\mathrm{op}}, \mathrm{Ab})$ is also abelian (kernels and cokernels are simply formed pointwise) and satisfies all of the above properties. We then deduce that the product preserving functors form an abelian subcategory, closed under all limits and colimits, and consequently satisfies the wanted properties as well. It remains to construct compact projective generators. For this, we note that the functor

$$\mathrm{Cond}_\kappa(\mathrm{Ab}) \to \mathrm{Cond}_\kappa(\mathrm{Set})$$

admits a left adjoint $X \mapsto \mathbb{Z}[X]$ (for instance by means of the adjoint functor theorem). Concretely, $\mathbb{Z}[X]$ is the sheafification of the presheaf

$$T \mapsto \mathbb{Z}[X(T)]$$

where $\mathbb{Z}[X(T)]$ is the free abelian group on the set $X(T)$. We claim that the collection of $\mathbb{Z}[\underline{T}]$, for $T \in \mathrm{edCH}_\kappa$ forms (a set of) compact projective generators. First, we note that

$$\mathrm{Hom}_{\mathrm{Cond}_\kappa(\mathrm{Ab})}(\mathbb{Z}[\underline{T}], F) = \mathrm{Hom}_{\mathrm{Cond}_\kappa(\mathrm{Set})}(\underline{T}, F) = F(T).$$

Therefore, $\mathrm{Hom}_{\mathrm{Cond}_\kappa(\mathrm{Ab})}(\mathbb{Z}[\underline{T}], -)$ commutes with all colimits of presheaves and therefore with 1-sifted colimits in $\mathrm{Cond}_\kappa(\mathrm{Ab})$. In particular, $\mathbb{Z}[\underline{T}]$ is compact projective. It now suffices to show that these objects form generators. Since $\mathrm{Cond}_\kappa(\mathrm{Ab})$ is abelian, it suffices to note that given a κ-condensed abelian group X with $X(T) = \mathrm{Hom}_{\mathrm{Cond}_\kappa}(\mathbb{Z}[\underline{T}], X) = 0$ for all $T \in \mathrm{edCH}_\kappa$, then $X = 0$.

Finally, we deduce that Cond(Ab) satisfies the same properties. We note that for $\kappa' > \kappa$, the canonical left adjoint $\mathrm{Cond}_\kappa(\mathrm{Ab}) \to \mathrm{Cond}_{\kappa'}(\mathrm{Ab})$ is again fully faithful, preserves colimits and limits, and is exact. The argument for fully faithfulness and preservation of colimits and limits is the same as in the case of κ- and κ'-condensed sets. One then finds that $\mathrm{Cond}_\kappa(\mathrm{Ab}) \subseteq \mathrm{Cond}_{\kappa'}(\mathrm{Ab})$ is an abelian subcategory just as we have argued in the case of $\mathrm{Cond}_\kappa(\mathrm{Ab})$ itself. The listed properties then follow for Cond(Ab) simply by observing that any diagram in Cond(Ab) is contained in $\mathrm{Cond}_\kappa(\mathrm{Ab})$ for a suitable κ, and then using that the properties hold in $\mathrm{Cond}_\kappa(\mathrm{Ab})$ as well as that the inclusion $\mathrm{Cond}_\kappa(\mathrm{Ab}) \to \mathrm{Cond}(\mathrm{Ab})$ is fully faithful, preserves limits and colimits and is exact.

The only difference is that in the case of Cond(Ab) there is only a proper class of compact projective generators, namely $\mathbb{Z}[T]$ for $T \in \mathrm{edCH}$. □

5.2. **Remark** We spell out here that for $T \in \mathrm{edCH}$, the object $\mathbb{Z}[T]$ is projective in the abelian category Cond(Ab) in the usual sense of homological algebra: Given a morphism $f\colon X \to Y$ in Cond(Ab) with trivial cokernel (i.e. a surjection in the previous sense), and a map $\mathbb{Z}[T] \to Y$, then there exists a lift $\mathbb{Z}[T] \to X$ along f. Indeed, we simply show that the map $X(T) \to Y(T)$ is surjective. So pick an element $y \in Y(T)$. Since f is surjective, there exists a cover $\{T_i \xrightarrow{\alpha_i} T\}$ such that $\alpha_i^*(y)$ lifts to an element $x_i \in X(T_i)$. Let $p\colon \amalg T_i \to T$ be the induced map. Consider then the diagram

$$\begin{tikzcd}
X(T) \arrow[r, "p^*"] \arrow[d, "f"] & X(\amalg T_i) \arrow[r, "\cong"] \arrow[d, "f"] & \prod_i X(T_i) \arrow[d] \\
Y(T) \arrow[r, "p^*"] & Y(\amalg_i T_i) \arrow[r, "\cong"] & \prod_i Y(T_i)
\end{tikzcd}$$

and observe that the family of elements (x_i) determine an element x of $X(\amalg T_i)$ whose image under f is given by $p^*(y)$. Since p is surjective, it admits a section $s\colon T \to \amalg T_i$. It then follows that

$$f(s^*(x)) = s^*(f(x)) = s^*(p^*(y)) = (ps)^*(y) = y.$$

5.3. **Remark** We will argue momentarily that every κ-condensed abelian group is a quotient of an object of the form $\bigoplus_{i\in I} \mathbb{Z}[T_i]$. Consequently, the smallest subcategory of $\mathrm{Cond}_\kappa(\mathrm{Ab})$ closed under colimits and containing the objects $\mathbb{Z}[T]$ for $T \in \mathrm{edCH}_\kappa$, is all of $\mathrm{Cond}_\kappa(\mathrm{Ab})$. This better justifies the terminology of *generators*. The "consequently" is obtained by considering a surjection $f\colon \bigoplus \mathbb{Z}[T_i] \to M$, and a further surjection $\bigoplus \mathbb{Z}[S_j] \to \ker(f)$. Then, we find that

$$\mathrm{coker}\big(\bigoplus \mathbb{Z}[S_j] \longrightarrow \bigoplus \mathbb{Z}[T_i]\big) \cong M.$$

Now we argue that any κ-condensed abelian group M admits such a surjection. By Zorn's lemma, there is a maximal sub object $M' \subseteq M$ which admits such a surjection. Indeed, for this, it suffices to show that for a totally ordered chain of sub objects $M_i \subseteq M$, such that each M_i admits a surjection from a sum of $\mathbb{Z}[T_j]$'s, then the colimit over M_i also admits such a surjection (an argument is required here). We claim that $M' = M$. Else, there is a surjection $M \to M/M' \neq 0$, and therefore there exists $S \in \mathrm{edCH}$ such that $M/M'(S) \neq 0$, so that there exists a non-zero map $\mathbb{Z}[S] \to M/M'$. Since $M \to M/M'$ is surjective and $\mathbb{Z}[S]$ is projective, this can be lifted to map $\mathbb{Z}[S] \to M$. The image of the map

$$\mathbb{Z}[S] \oplus \bigoplus_I \mathbb{Z}[T_i] \to M \oplus M' \xrightarrow{+} M$$

is then strictly bigger than M', contradicting the maximality of M'.

5.4. **Remark** It is general non-sense that the category $\mathrm{Cond}_\kappa(\mathrm{Ab})$ of abelian group objects in the topos $\mathrm{Cond}_\kappa(\mathrm{Set})$ is an abelian category, satisfying some, but not all of the above axioms. More specifically, $\mathrm{Sh}(\mathcal{C}; \mathrm{Ab})$ is always an abelian category with all small colimits and limits and where filtered colimits are exact and has enough injectives. In general, it need not contain compact projective generators (in fact, in many cases that appear in nature, i.e. in algebraic geometry, it does not), and (infinite) products/coproducts need not in general be exact.

To obtain all the properties appearing in Theorem 5.1, we again heavily use that κ-condensed abelian groups are very close to a diagram category, namely product preserving abelian group valued functors on the (opposite of the) category of κ-small extremally disconnected compact Hausdorff spaces.

We add further properties of the categories $\mathrm{Cond}_\kappa(\mathrm{Ab})$ and $\mathrm{Cond}(\mathrm{Ab})$.

5.5. **Proposition** *The category* $\mathrm{Cond}(\mathrm{Ab})$ *is equipped with a closed symmetric monoidal tensor product for which the functor* $\mathbb{Z}[-]\colon \mathrm{CH} \to \mathrm{Cond}(\mathrm{Ab})$ *is symmetric monoidal. The objects* $\mathbb{Z}[T]$ *are flat, i.e. the functor* $\mathbb{Z}[T] \otimes -$ *is exact.*

Proof. We work with fixed κ-condensed abelian groups first and show that the left Kan extensions $\mathrm{Cond}_\kappa(\mathrm{Ab}) \to \mathrm{Cond}_{\kappa'}(\mathrm{Ab})$ are canonically symmetric monoidal. Then $\mathrm{Cond}(\mathrm{Ab})$ inherits a symmetric monoidal structure for which all inclusions $\mathrm{Cond}_\kappa(\mathrm{Ab}) \subseteq \mathrm{Cond}(\mathrm{Ab})$ are symmetric monoidal. We define the tensor product on $\mathrm{Cond}_\kappa(\mathrm{Ab})$ as follows. Let $M, N \in \mathrm{Cond}_\kappa(\mathrm{Ab})$. Then $M \otimes N$ is the sheafification of the presheaf

$$T \mapsto M(T) \otimes_{\mathbb{Z}} N(T).$$

We then have that $\mathbb{Z}[S] \otimes \mathbb{Z}[S']$ is the sheafification of the presheaf

$$T \mapsto \mathbb{Z}[\mathrm{Map}(T,S)] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{Map}(T,S')] \cong \mathbb{Z}[\mathrm{Map}(T,S) \times \mathrm{Map}(T,S')] \cong \mathbb{Z}[\mathrm{Map}(T, S \times S')]$$

and therefore, we have $\mathbb{Z}[S] \otimes \mathbb{Z}[S'] = \mathbb{Z}[S \times S']$. Furthermore, an exercise below says that the functor $M \otimes -$ preserves colimits.

Now we define an internal Hom object as follows. For $T \in \mathrm{edCH}$, we set

$$\underline{\mathrm{Hom}}(N, N')(T) = \mathrm{Hom}(\mathbb{Z}[T] \otimes N, N').$$

We note that this is a condensed abelian group since $\mathbb{Z}[T \amalg T'] = \mathbb{Z}[T] \oplus \mathbb{Z}[T']$. This follows from **??**. Since we have

$$\underline{\mathrm{Hom}}(N, N')(T) = \mathrm{Hom}(\mathbb{Z}[T], \underline{\mathrm{Hom}}(N, N'))$$

we find a special case of the needed adjunction bijection. It is then formal to see (use that any condensed abelian group is a cokernel of sums of condensed abelian groups of the form $\mathbb{Z}[T]$ and that the tensor product preserves colimits) that we have an adjunction bijection

$$\mathrm{Hom}(M \otimes N, N') \cong \mathrm{Hom}(M, \underline{\mathrm{Hom}}(N, N'))$$

for general M as needed.

Next, we argue that $\mathbb{Z}[T]$ is flat. To do so, consider an exact sequence of condensed abelian groups. Then the tensor product of wit $\mathbb{Z}[T]$ can be computed by taking the presheaf tensor product with the presheaf $S \mapsto \mathbb{Z}[T(S)]$ of which $\mathbb{Z}[T]$ is the sheafification, and then sheafifying everything. Since sheafification commutes with colimits and finite limits, it is an exact functor. Therefore, it suffices to show that the presheaf tensor product is again exact. This follows from the fact that $\mathbb{Z}[T(S)]$ is a free and hence flat abelian group.

Now we need to show that everything is compatible with enlarging κ. First, we argue that the functor

$$\mathrm{Cond}_\kappa(\mathrm{Ab}) \to \mathrm{Cond}_{\kappa'}(\mathrm{Ab})$$

is symmetric monoidal. Since every object is a cokernel of objects of the form $\mathbb{Z}[T]$, this follows from the fact that the functor $\mathrm{CH}_\kappa \to \mathrm{CH}_{\kappa'}$ is symmetric monoidal. We deduce that $\mathrm{Cond}(\mathrm{Ab})$ is canonically symmetric monoidal. Moreover, the functor $\mathrm{CH} \to \mathrm{Cond}(\mathrm{Ab})$ is then also symmetric monoidal (this is an argument similar to the one appearing in **??**).

To see that it is closed symmetric monoidal, it would suffice to prove that

$$\iota\colon \mathrm{Cond}_\kappa(\mathrm{Ab}) \to \mathrm{Cond}_{\kappa'}(\mathrm{Ab}) \tag{3}$$

also preserves the internal hom. However, I don't think this is the case, and indeed something weaker is sufficient: Namely, we need to show that for fixed $M, N \in \mathrm{Cond}(\mathrm{Ab})$, there is a sufficiently large κ such that for $\kappa' \geq \kappa$ the canonical map

$$\iota(\underline{\mathrm{Hom}}_\kappa(M, N)) \to \underline{\mathrm{Hom}}_{\kappa'}(M, N)$$

is an isomorphism. Once this is so, it follows that this is also the internal hom in Cond(Ab) for formal reasons. So it is this property that we shall argue.

We first note that since the symmetric monoidal structure on $\mathrm{Cond}_\kappa(\mathrm{Ab})$ preserves colimits in each variable, we find that for any diagram $M \colon I \to \mathrm{Cond}_\kappa(\mathrm{Ab})$, the canonical map

$$\underline{\mathrm{Hom}}_\kappa(\operatorname*{colim}_{i \in I} M_i, N) \to \lim_{i \in I} \underline{\mathrm{Hom}}_\kappa(M_i, N)$$

is an isomorphism. Indeed, this follows from the Yoneda lemma.

Now, we first choose τ such that $M, N \in \mathrm{Cond}_\tau(\mathrm{Ab})$. Then we write

$$M = \mathrm{coker}(\bigoplus_I \mathbb{Z}[T_i] \to \bigoplus_J \mathbb{Z}[S_j]),$$

with T_i and S_j objects of edCH_τ. Then we may choose κ large enough that the cardinalities of I and J are smaller than the cofinality of κ. In this case, for $\kappa' \geq \kappa$, the functor $\iota \colon \mathrm{Cond}_\kappa(\mathrm{Ab}) \to \mathrm{Cond}_{\kappa'}(\mathrm{Ab})$ preserves colimits and limits indexed over I and J. It follows that it suffices to prove that the map (3) is an isomorphism in the case $M = \mathbb{Z}[T]$ again with $T \in \mathrm{edCH}_\tau$. Now, by Proposition 4.32, we may find a strong limit cardinal κ with $\mathrm{cof}(\kappa) = \lambda$ such that $N \in \mathrm{Cond}_\kappa(\mathrm{Set})$ and such that N transforms λ-cofiltered limits to λ-filtered colimits (and then likewise for all larger κ's). Then we choose κ so large that $\mathbb{Z}[T] \in \mathrm{Cond}_\kappa(\mathrm{Ab})$ and N satisfies this colimit property and again let $\lambda = \mathrm{cof}(\kappa)$. Spelling out the definitions, to show that the map in (3) is an isomorphism, we obtain the map

$$\operatorname*{colim}_{S' \in (\mathrm{edCH}_\kappa^{\mathrm{op}})_{/S}} N(S' \times T) \longrightarrow N(S \times T).$$

where the colimit ranges through all maps $S \to S'$ with $S' \in \mathrm{edCH}_\kappa$. Now, this slice category is λ-filtered as we have argued in Theorem 4.14, and so the map above is isomorphic to the map

$$N(\lim_{S \to S'} S' \times T) \to N(S \times T)$$

induced by the canonical map $S \to \lim_{S \to S'} S'$. This map is a homeomorphism so we conclude the proof of the proposition. $\square$

Exercise. Show that the functor $M \otimes - \colon \mathrm{Cond}_\kappa(\mathrm{Ab}) \to \mathrm{Cond}_\kappa(\mathrm{Ab})$ preserves colimits.

Proof. To see this, let $N = L(\operatorname{colim} N_i)$ be a colimit of κ-condensed abelian groups. Here, the colimit is formed in presheaves and L denotes the sheafification functor. We claim that the canonical map of presheaves sending T to the map

$$M(T) \otimes \operatorname{colim} N_i(T) \to M(T) \otimes L(\operatorname{colim} N_i(T))$$

is an equivalence after applying the sheafification functor L. In other words, we need to show that for each condensed abelian group A, the canonical map

$$\mathrm{Hom}(M \otimes L(\operatorname{colim} N_i), A) \to \mathrm{Hom}(M \otimes \operatorname{colim} N_i, A)$$

is an isomorphism. To do so, we note that the tensor product corepresents bilinear maps out of the product, i.e. we have

$$\mathrm{Hom}(M \otimes M', A) = \mathrm{Hom}^{\mathrm{bil}}(M \times M', A).$$

Applying this for both sides of the above morphism (i.e. for $M' = L(\operatorname{colim} N_i)$ and $M' = \operatorname{colim} N_i$ and the canonical map between them), and using that sheafification commutes with finite products, the claim follows. $\square$

5.1. **Stable ∞-categories and derived categories.** In what follows, we will use the notion of ∞-categories – the reader should be fine pretending that limits and colimits in ∞-categories behave the way we know from ordinary category theory. The ordinary category of sets is then always replaced by the ∞-category of animae, written An and we should think that, just as Set is the free cocompletion of $*$ in ordinary categories (i.e. Set is the category we obtain by closing $*$ up by all small colimits), An is the free cocompletion of $*$ in ∞-categories. That is, the universal property of An is the following: There is a canonical functor $* \to \mathrm{An}$ which, for any cocomplete ∞-category $\mathcal{C}$ induces the following equivalence of categories:

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{An}, \mathcal{C}) \xrightarrow{\simeq} \mathcal{C}.$$

More generally, for $\mathcal{D}$ a small ∞-category (replacing $*$) we have

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{An}), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{D}, \mathcal{C}).$$

5.6. **Definition** For an ∞-category $\mathcal{C}$, there is a canonical functor

$$\mathrm{Map}_{\mathcal{C}}(-,) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$$

sending a pair of objects X, Y the the anima of maps between them.

The following is the analog of the usual Yoneda yoga:

5.7. **Theorem** *The resulting functor*

$$\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$$

is fully faithful and preserves limits (and limits in functor categories are formed pointwise).

5.8. **Remark** We then think of $\pi_0\mathrm{Map}_{\mathcal{C}}(X, Y)$ as homotopy classes of maps from X to Y. The objects of $\mathcal{C}$, together with these homotopy classes of maps form an ordinary category, the *homotopy category* $h\mathcal{C}$ associated to $\mathcal{C}$.

We now need a number of definitions.

5.9. **Definition** An ∞-category is called

(1) pointed, if it admits initial and terminal objects $\varnothing$ and $*$, and the canonical map $\varnothing \to *$ is an equivalence,
(2) preadditive, if it admits finite coproducts and finite products, and the canonical map $\amalg_i X_i \to \prod_i X_i$ is an equivalence,
(3) additive, if it is preadditive and for all objects X, the shear map $X \oplus X \to X \oplus X$ is an equivalence,
(4) stable, if it admits finite limits and finite colimits, and a commutative square is a pushout if and only if it is a pullback.

Exercise. Let $\mathcal{C}$ be a preadditive ∞-category. Show that the hom sets in $h\mathcal{C}$ are naturally abelian monoids. Show furthermore that if $\mathcal{C}$ is additive, then these abelian monoids are in fact abelian groups.

Exercise. Show that a stable category is additive.

5.10. **Remark** In a stable category, for every object, we may form the pushout of the two canonical maps $0 \leftarrow X \to 0$, and denote this by $X[1]$. This refines to an endofunctor [1] called the shift operator. This functor is an equivalence, with inverse given by the pullback of the diagram $0 \to X \leftarrow 0$, denoted then by $X[-1]$. Iterating these functors given a functor $X \mapsto X[n]$ for all $n \in \mathbb{Z}$.

Exercise. Show that $[-1]$ is indeed inverse to $[1]$.

5.11. **Remark** Let $\mathcal{C}$ be a stable category. For objects $X, Y \in \mathcal{C}$ we write $[X, Y]_d = \pi_0 \mathrm{Map}_{\mathcal{C}}(X[d], Y) \cong \pi_d(\mathrm{Map}_{\mathcal{C}}(X, Y))$. Now, a fibre sequence in $\mathcal{C}$, i.e. a pullback diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

gives rise to a long exact sequence

$$\dots \to [A, X]_d \to [A, Y]_d \to [A, Z]_d \to [A, X]_{d-1} \to \dots$$

simply because the functor $\mathrm{Map}_{\mathcal{C}}(A, -) \colon \mathcal{C} \to \mathrm{An}$ preserves limits and fibre sequences in anima give rise to long exact sequences of homotopy groups. Dually, since the above fibre sequence is also a cofibre sequence, we also obtain a long exact sequence

$$\dots \to [Z, A]_d \to [Y, A]_d \to [X, A]_d \to [Z, A]_{d-1} \to \dots$$

since the functor $\mathrm{Map}_{\mathcal{C}}(-, A) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ preserves limits (i.e. sends colimits in $\mathcal{C}$ to limits in An).

The main example we care about in this lecture are derived categories of rings, and actually first and foremost the derived category $\mathcal{D}(\mathbb{Z})$ of $\mathbb{Z}$. We will follow an approach we learned from Dustin Clausen and introduce the derived ∞-category of a ring in an axiomatic way. The uniqueness part of the following result is known as the Schwede–Shipley theorem and we simply take the existence part for granted for the moment – we will in fact give an existence argument later as well.

5.12. **Proposition** *Let R be a ring. There exists a unique cocomplete stable ∞-category, written $\mathcal{D}(R)$, equipped with a distinguished compact object R and an identification of rings $\pi_0(\mathrm{End}_{\mathcal{D}(R)}(R)) \cong R$, such that $[R, R]_n = 0$ for $n \neq 0$ and such that $\mathcal{D}(R)$ is generated (as a stable category) under colimits by R.*

5.13. **Remark** That $\mathcal{D}(R)$ is generated as a stable category under colimits by R simply means that if $\mathcal{C} \subseteq \mathcal{D}(R)$ is a stable subcategory (i.e. a full subcategory closed under finite limits and colimits in $\mathcal{D}(R)$) which is cocomplete and contains R, then $\mathcal{C} = \mathcal{D}(R)$.

5.14. **Definition** We define homology functors $H_n \colon \mathcal{D}(R) \to \mathrm{Ab}$ by $H_n(X) = [R, X]_n = [R[n], X]_0$.

5.15. **Remark** We note that $H_n(X)$ in fact is naturally an R-module, i.e. an object of $\mathrm{Mod}(R)$. More precisely, the homology functors are in fact functors $\mathcal{D}(R) \to \mathrm{Mod}(R)$. Indeed, we need to construct canonical maps

$$H_n(X) \times R \to H_n(X)$$

which implement the right R-module structure. To do so, we note that $R \cong [R,R]_0 \cong [R[n],R[n]]_0$, so that the map

$$[R[n],X]_0 \times [R[n],R[n]]_0 \to [R[n],X]$$

can be defined by precomposition of maps. One then checks that this indeed provides a functorial R-module structure on $H_n(X)$.

5.16. **Remark** By Remark 5.11, we find that a fibre sequence, i.e. a pullback diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

in $\mathcal{D}(R)$, gives rise to a long exact sequence of homology groups.

5.17. **Lemma** *A map $f\colon X \to Y$ is an equivalence if and only if it induces an isomorphism on all homology groups.*

Proof. First, f is an equivalence if and only if its cofibre vanishes (exercise). Next, we argue that the cofibre vanishes if and only if all its homology groups are trivial, essentially because R generates $\mathcal{D}(R)$ (as a stable category) under colimits. In more detail, fix an object $X \in \mathcal{D}(R)$ and let $\mathcal{C} \subseteq \mathcal{D}(R)$ be the full subcategory consisting of objects such that $[Y,X]_n = 0$ for all $n \in \mathbb{Z}$. Then $\mathcal{C}$ is a stable subcategory of $\mathcal{D}(R)$ and contains R if the homology groups of X vanish. Therefore, we conclude that an object is trivial if and only if all of its homology groups vanish. The result then follows from the long exact sequence of homology groups just mentioned. □

For the following Lemma, we take for granted that the category $\mathcal{D}(R)$ has all small limits. We could simply add this to our defining properties of $\mathcal{D}(R)$ for now - but a posteriori it can be shown that this is not needed: $\mathcal{D}(R)$ turns out to be a complete (in fact presentable) category.

5.18. **Lemma** *The homology functors $H_d\colon \mathcal{D}(R) \to \mathrm{Mod}(R)$ commute with products, coproducts, and filtered colimits. For sequential inverse limits, there is a lim-lim1-sequence "the Milnor sequence".*

Proof. One checks that it commutes with products (here we use that taking π_0 commutes with products). It commutes with filtered colimits since we required R to be a compact object. Then, since finite coproducts are products, and general coproducts are filtered colimits of finite coproducts, homology also commutes with coproducts. For the Milnor sequence, we use that a sequential inverse limit $\{X_n\}_{n\in\mathbb{Z}}$ can equivalently be described as a fibre of products (as a consequence of writing a general limit as an equaliser of products and then using that equalisers of two maps are given by the fibre of their difference). The outcome is a fibre sequence

$$\lim_{n\in\mathbb{Z}} X_n \longrightarrow \prod_{n\in\mathbb{Z}} X_n \stackrel{\mathrm{id}-(g_n)}{\longrightarrow} \prod_{n\in\mathbb{Z}} X_n$$

where the map (g_n) is given by the family of maps $\prod X_n \to X_{n+1} \to X_n$ where the first map is the canonical projection and the second map is the structure map of the sequential diagram. Using the commutation of homology with products we obtain the following long exact sequence

$$\cdots \to \prod_{n\in\mathbb{Z}} H_{k+1}(X_n) \stackrel{\mathrm{id}-(g_n)_*}{\to} \prod_{n\in\mathbb{Z}} H_{k+1}(X_n) \to H_k(\lim_{n\in\mathbb{Z}} X_n) \to \prod_{n\in\mathbb{Z}} H_k(X_n) \stackrel{\mathrm{id}-(g_n)_*}{\to} \prod_{n\in\mathbb{Z}} H_k(X_n) \to \dots$$

The cokernel of the first map appearing in the above sequence is given by $\lim^1_{n\in\mathbb{Z}} H_{k+1}(X_n)$ and the kernel of the last map is given by $\lim_{n\in\mathbb{Z}} H_k(X_n)$. Therefore, we obtain a short exact sequence

$$0 \to \lim_{n\in\mathbb{Z}}{}^1 H_{k+1}(X_n) \to H_k(\lim_{n\in\mathbb{Z}} X_n) \to \lim_{n\in\mathbb{Z}} H_k(X_n) \to 0$$

called the Milnor sequence, or also the lim-lim1-sequence. $\square$

5.19. **Definition** Let $\mathcal{D}(R)_{\geq 0}$ and $\mathcal{D}(R)_{\leq 0}$ be the full subcategories on objects X with $H_d(X) = 0$ for $d < 0$ and $d > 0$, respectively.

5.20. **Proposition** *Let $X \in \mathcal{D}(R)$. Then there exists $Y \in \mathcal{D}(R)_{\geq 0}$ and a map $f\colon Y \to X$ such that f induces an isomorphism on H_d for $d \geq 0$.*

Proof. We inductively define a sequence of maps

$$Y_0 \to Y_1 \to Y_2 \to \dots$$

in $\mathcal{D}(R)_{/X}$ such that $Y_i \in \mathcal{D}(R)_{\geq 0}$ and $Y_i \to X$ is an isomorphism on H_d for $0 \leq d \leq i-1$ and surjective on H_i. For Y_0 we consider $\bigoplus_{x\in H_0(X)} R$ with its canonical map to X. This map is surjective on H_0 by construction. Now, to define Y_{n+1} from Y_n, we consider the fibre sequence

$$F \to Y_n \to X.$$

By construction, $F \in \mathcal{D}(R)_{\geq n}$ and $H_n(F)$ surjects onto the kernel of the map $H_n(Y_n) \to H_n(X)$. We can then find a map $\bigoplus_I R[n] \to F$ such that the composite

$$\bigoplus_I R[n] \to F \to Y_n$$

surjects on H_n onto the kernel of $H_n(Y_n) \to H_n(X)$. Define Y_{n+1} as the cofibre of the above composite, and extend the map from $Y_n \to X$ over Y_{n+1}. Then $Y = \operatorname{colim}_n Y_n$ satisfies all properties. $\square$

5.21. **Lemma** *The following conditions on an object $X \in \mathcal{D}(R)$ are equivalent:*

(1) $X \in \mathcal{D}(R)_{\geq 0}$,
(2) *X lies in the smallest cocomplete subcategory (not stable!) generated by R, and*
(3) *there is a sequence of maps $X_0 \to X_1 \to \dots$ with colimit X and such that* $\mathrm{cofib}(X_{n-1} \to X_n)$ *is equivalent to $\bigoplus_I R[n]$ for some set I (here we set $X_{-1} = 0$).*

Proof. That (1) implies (3) follows from the proof of the above proposition (and Lemma 5.17). (3) implies (2) inductively, since $X_n = \mathrm{cofib}(\bigoplus_I R[n-1] \to X_{n-1})$. (2) implies (1) simply because $R \in \mathcal{D}(R)$ and $\mathcal{D}(R)$ is closed under colimits (Exercise). $\square$

5.22. **Corollary** *Let $X \in \mathcal{D}(R)_{\geq 0}$ and $Y \in \mathcal{D}(R)_{\leq -1}$. Then $\mathrm{Map}_{\mathcal{D}(R)}(X, Y) \simeq *$.*

Proof. We have $\mathrm{map}(R, Y) = *$ since $Y \in \mathcal{D}(R)_{\leq -1}$. Now, the collection of objects $X \in \mathcal{D}(R)$ for which $\mathrm{map}(X, Y) \simeq *$ is closed under colimits (because limits of contractible anima are contractible) and contains R. Therefore, it contains $\mathcal{D}(R)_{\geq 0}$ by the above remark. $\square$

5.23. **Corollary** *The inclusion $\mathcal{D}(R)_{\geq 0} \subseteq \mathcal{D}(R)$ admits a right adjoint $\tau_{\geq 0}$. The inclusion $\mathcal{D}(R)_{\leq 0} \subseteq \mathcal{D}(R)$ admits a left adjoint $\tau_{\leq 0}$. The pair $(\mathcal{D}(R)_{\geq 0}, \mathcal{D}(R)_{\leq 0})$ therefore forms what is called a t-structure on $\mathcal{D}(R)$.*

Proof. As in ordinary category theory, to show that $\tau_{\geq 0}$ exists, it suffices to show that for each object X, we can find $Y \in \mathcal{D}(R)_{\geq 0}$ and a map $Y \to X$ in $\mathcal{D}(R)$, such that for all $Z \in \mathcal{D}(R)_{\geq 0}$, the composite

$$\mathrm{Map}_{\mathcal{D}(R)_{\geq 0}}(Z, Y) \to \mathrm{Map}_{\mathcal{D}(R)}(Z, Y) \to \mathrm{Map}_{\mathcal{D}(R)}(Z, X)$$

is an equivalence. The first map is an equivalence since the inclusion is fully faithful by definition. Using a Y as in Proposition 5.20, we note that the cofibre C of $Y \to X$ is contained in $\mathcal{D}(R)_{\leq -1}$. Moreover, we have a fibre sequence

$$\mathrm{Map}_{\mathcal{D}(R)}(Z, Y) \to \mathrm{Map}_{\mathcal{D}(R)}(X, Y) \to \mathrm{Map}_{\mathcal{D}(R)}(C, X)$$

and the last term is contractible by Corollary 5.22. This gives that any Y as in Proposition 5.20 "is" $\tau_{\geq 0} X$. For $\tau_{\leq 0}$, we have to find $Z \in \mathcal{D}(R)_{\leq 0}$ and a map $X \to Z$ such that for all $A \in \mathcal{D}(R)_{\leq 0}$, the map

$$\mathrm{Map}_{\mathcal{D}(R)}(Z, A) \to \mathrm{Map}_{\mathcal{D}(R)}(X, A)$$

is an equivalence. We leave this part as an exercise. $\square$

5.24. **Remark** By shifting, we also find that for all $n \in \mathbb{Z}$, the inclusion $\mathcal{D}(R)_{\geq n} \subseteq \mathcal{D}(R)$ has a right adjoint $\tau_{\geq n}$ and the inclusion $\mathcal{D}(R)_{\leq n} \subseteq \mathcal{D}(R)$ has a left adjoint $\tau_{\leq n}$. They are connected by a fibre sequence of functors

$$\tau_{\geq n} \to \mathrm{id} \to \tau_{\leq n-1}.$$

5.25. **Remark** Observing the compatibility between the functors $\tau_{\geq n}$ for varying n, and likewise for $\tau_{\leq n}$, one obtains functorial towers

$$\dots \to \tau_{\leq n} X \to \tau_{\leq n-1} X \to \dots.$$

with inverse limit X and colimit 0, and likewise

$$\dots \to \tau_{\geq n} X \to \tau_{\geq n-1} X \to \dots$$

with colimit X and limit 0. Indeed, to check the claims about limit and colimit, look on homology and use that homology commutes with filtered colimits, and for the limits use the lim-lim1 sequence together with the Mittag–Leffler condition (all maps appearing in the lim1 diagram are surjective), showing that in each of the inverse limits, the lim1 term appearing (a priori) in homology vanishes.

5.26. **Definition** We let $\mathcal{D}(R)^{\heartsuit} = \mathcal{D}(R)_{\geq 0} \cap \mathcal{D}(R)_{\leq 0}$ denote the heart of the t-structure.

5.27. **Proposition** *The functor $H_0 \colon \mathcal{D}(R) \to \mathrm{Mod}(R)$ induces an equivalence of ∞-categories $\mathcal{D}(R)^{\heartsuit} \simeq \mathrm{Mod}(R)$.*

Proof. Corollary 5.22 implies that $\mathcal{D}(R)^{\heartsuit}$ is an ordinary category, i.e. that for $X, Y \in \mathcal{D}(R)^{\heartsuit}$, we have $[X,Y]_d = 0$ unless $d = 0$. We also observe that $\mathcal{D}(R)^{\heartsuit}$ is an abelian category: the kernel of a map f is given by $\tau_{\geq 0}\mathrm{fib}(f)$, and the cokernel is given by $\tau_{\leq 0}\mathrm{cofib}(f)$. Consequently, $H_0 \colon \mathcal{D}(R)^{\heartsuit} \to \mathrm{Mod}(R)$ is an exact functor of abelian categories. Moreover, it commutes with coproducts as we have seen earlier. We deduce that H_0 is essentially surjective, by picking a presentation of an arbitrary module, i.e. writing it as a cokernel between direct sums of R's. We now show that $R \in \mathcal{D}(R)^{\heartsuit}$ is compact and projective - as is $R \in \mathrm{Mod}(R)$. Indeed, it is compact by assumption in $\mathcal{D}(R)$, it is then formal to see that it is also compact in $\mathcal{D}(R)^{\heartsuit}$ since the inclusion $\mathcal{D}(R)^{\heartsuit} \to \mathcal{D}(R)$ commutes with filtered colimits (since homology does). To see that it is projective, let $X \to Y$ be a surjection in the abelian category $\mathcal{D}(R)^{\heartsuit}$. To see that R is projective, it suffices to show that $H_0(X) \to H_0(Y)$ is surjective, which follows from the earlier observation that H_0 is exact.

This implies that the functor $H_0\mathcal{D}(R)^{\heartsuit} \to \mathrm{Mod}(R)$ is fully faithful: Indeed, let us fix $M \in \mathcal{D}(R)^{\heartsuit}$ and consider the full subcategory of $\mathcal{D}(R)^{\heartsuit}$ consisting of objects N such that the map

$$\mathrm{Hom}(N, M) \to \mathrm{Hom}(H_0(N), H_0(M))$$

is an isomorphism. Then this subcategory is closed under colimits and contains R. Since $\mathcal{D}(R)_{\geq 0}$ is generated under colimits by R and $\tau_{\leq 0} \colon \mathcal{D}(R)_{\geq 0} \to \mathcal{D}(R)^{\heartsuit}$ is a left adjoint, we deduce that R generates $\mathcal{D}(R)_{\geq 0}$ under colimits as well. Consequently, we deduce that for all $N \in \mathcal{D}(R)^{\heartsuit}$, the map above is an isomorphism, showing that $H_0 \colon \mathcal{D}(R)^{\heartsuit} \to \mathrm{Mod}(R)$ is fully faithful as desired. $\square$

We will always use the above equivalence to implicitly think of objects of $\mathrm{Mod}(R)$ as objects of $\mathcal{D}(R)$. The above suggests that studying maps between objects of $\mathrm{Mod}(R)$ in the derived category $\mathcal{D}(R)$ is a reasonable thing to do. Here is another instance of this:

5.28. **Proposition** *Let $M, N \in \mathrm{Mod}(R)$ and $d \geq 0$. We have a functorial equivalence*

$$[M, N]_{-d} \cong \mathrm{Ext}^d_R(M, N).$$

Proof. We simply need to show that for fixed N, the functor $[-, N]_*$ is the derived functor of $\mathrm{Hom}_R(-, N)$. For this, one shows that it is a universal δ-functor, we refrain from spelling out the details here (but recommend doing so as an exercise) but note that a useful thing to observe here is that a δ-functor is universal if its positive values vanish on projectives. $\square$

Finally, if one already knows about the derived category from some other course, we expect there to be a relation between the derived category and a category of chain complexes of R-modules. In the sequel, we construct a functor $\mathrm{Ch}(\mathrm{Mod}(R)) \to \mathcal{D}(R)$. For this, we shall take a small detour.

5.29. **Definition** We denote by $\mathrm{Fil}(\mathcal{D}(R)) = \mathrm{Fun}(\mathbb{Z}, \mathcal{D}(R))$ the category of filtered objects in $\mathcal{D}(R)$. Here, we think of $\mathbb{Z}$ as a poset under the relation $\leq$. A filtered object F is called convergent if $F(-\infty) = \lim F(n) = 0$ and in this case we say it converges to $\operatorname{colim} F(n) = F(\infty)$. The n'th associated graded of a filtered object F is given by $\mathrm{gr}^n(F) = \mathrm{cofib}(F(n-1) \to F(n))$, sometimes also written $F(n)/F(n-1)$.

5.30. **Remark** A morphism from F to G in $\mathrm{Fil}(\mathcal{D}(R))$ consists of commutative squares

$$\begin{tikzcd} \dots \arrow[r] & F(n) \arrow[r] \arrow[d] & F(n+1) \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & G(n) \arrow[r] & G(n+1) \arrow[r] & \dots \end{tikzcd}$$

with no further coherence between them.

5.31. **Remark** Associating to a filtered object its associated graded induces a functor

$$\mathrm{Fil}(\mathcal{D}(R)) \to \mathrm{gr}(\mathcal{D}(R))$$

where $\mathcal{D}(R) = \mathrm{Fun}(\mathbb{Z}^\delta, \mathcal{D}(R))$. This functor is conservative on convergent objects. To show this, it suffices to know that a convergent object F with trivial associated graded is trivial. Since the associated graded is trivial, we find that for all $n \in \mathbb{Z}$, the map $F(n-1) \to F(n)$ is an equivalence. This implies that $F(n) \simeq \lim_n F(n) = 0$ for all $n \in \mathbb{Z}$ as needed.

Exercise. Show that the above functor is not conservative in general.

5.32. **Remark** For $F \in \mathrm{Fil}(\mathcal{D}(R))$, there is a canonical map

$$\mathrm{gr}^{n+1}(F) \to \mathrm{gr}^n(F)[1]$$

induced by the diagram consisting of pushout squares

$$\begin{tikzcd} F(n-1) \arrow[r] \arrow[d] & F(n) \arrow[r] \arrow[d] & F(n+1) \arrow[d] \\ 0 \arrow[r] & \mathrm{gr}^n(F) \arrow[r] \arrow[d] & F(n+1)/F(n-1) \arrow[d] \\ & 0 \arrow[r] & \mathrm{gr}^{n+1}(F) \end{tikzcd}$$

or more precisely, by once rotating the lower cofibre sequence.

5.33. **Definition** We denote by $\mathrm{Fil}(\mathcal{D}(R))^\heartsuit$ the full subcategory of $\mathrm{Fil}(\mathcal{D}(R))$ on convergent objects F such that $\mathrm{gr}^n(F)[-n] \in \mathcal{D}(R)^\heartsuit$ for all $n \in \mathbb{Z}$.

5.34. **Remark** As the notation suggests, $\mathrm{Fil}(\mathcal{D}(R))^\heartsuit$ is related to the heart of a t-structure on $\mathrm{Fil}(\mathcal{D}(R))$. Indeed, a t-structure is given as follows

(1) $\mathrm{Fil}(\mathcal{D}(R))_{\geq 0} = \{F \in \mathrm{Fil}(\mathcal{D}(R)) \mid \forall n \in \mathbb{Z},\ F(n) \in \mathcal{D}(R)_{\geq n}\}$, and
(2) $\mathrm{Fil}(\mathcal{D}(R))_{\leq 0} = \{F \in \mathrm{Fil}(\mathcal{D}(R)) \mid \forall n \in \mathbb{Z},\ F(n) \in \mathcal{D}(R)_{\leq n}\}$.

This t-structure is often referred to as the Beilinson t-structure on filtered objects. Observe that an object in $\mathrm{Fil}(\mathcal{D}(R))^{\heartsuit}$ satisfies that $\mathrm{gr}^n(F)[-n] \in \mathcal{D}(R)^{\heartsuit}$: Indeed, there is a fibre sequence

$$F(n)[-n] \to \mathrm{gr}^n(F)[-n] \to F(n-1)[-(n-1)]$$

and both $F(n)[-n]$ and $F(n-1)[-(n-1)]$ are objects of $\mathcal{D}(R)^{\heartsuit}$, hence so is $\mathrm{gr}^n(F)[-n]$. We claim that the converse is true provided F is convergent, but not in general.

Exercise. Let $F \in \mathrm{Fil}(\mathcal{D}(R))_{\geq 0}$ and $G \in \mathrm{Fil}(\mathcal{D}(R))_{\leq -1}$. Show that $\mathrm{Map}(F, G) \simeq *$.

5.35. **Definition** For $F \in \mathrm{Fil}(\mathcal{D}(R))^{\heartsuit}$, we form the following: Set $M_n = H_n(\mathrm{gr}^n(F)) \in \mathrm{Mod}(R)$ and define a map $d_n \colon M_n \to M_{n-1}$ by the map induced on H_n by the map

$$\mathrm{gr}^n(F) \to \mathrm{gr}^{n-1}(F)[1]$$

from Remark 5.32.

5.36. **Proposition** *For $F \in \mathrm{Fil}(\mathcal{D}(R))^{\heartsuit}$ and (M_n, d_n) as above we find that $d^2 = 0$, thus (M_n, d_n) defines a chain complex in $\mathrm{Mod}(R)$. The resulting functor $\mathrm{Fil}(\mathcal{D}(R))^{\heartsuit} \to \mathrm{Ch}(\mathrm{Mod}(R))$ is an equivalence of ∞-categories. Moreover, $H_n(F(\infty))$ and $H_n(M_n, d_n)$ are canonically isomorphic.*

Proof. First we show that the required composite is indeed 0. For this, it is more understandable to write $\mathrm{gr}^n(F)$ as the quotient $\frac{F(n)}{F(n-1)}$. Doing so, we recall that there is a fibre sequence

$$\tfrac{F(n)}{F(n-2)}[1] \to \tfrac{F(n)}{F(n-1)}[1] \to \tfrac{F(n-1)}{F(n-2)}[2].$$

Therefore, to see that the composite

$$\tfrac{F(n+1)}{F(n)} \to \tfrac{F(n)}{F(n-1)}[1] \to \tfrac{F(n-1)}{F(n-2)}[2]$$

is trivial, it suffices to observe that the first map canonically factors as

$$\tfrac{F(n+1)}{F(n)} \to \tfrac{F(n)}{F(n-2)}[1] \to \tfrac{F(n)}{F(n-1)}[1].$$

Then, we show by induction that the above construction induces an equivalence (compatible with taking homology of the underlying object of the LHS and the chain complex on the right hand side), for all $n \leq m$

$$\mathrm{Fil}(\mathcal{D}(R))^{\heartsuit}_{[n,m]} \to \mathrm{Ch}(\mathrm{Mod}(R))_{[n,m]}$$

where the subscript on the left denotes the full subcategory on filtrations F with $F(k) = F(n-1)$ for all $k \leq n-1$ and $F(k) = F(m)$ for all $k \geq m$, and on the right hand side denotes chain complexes with trivial terms outside the interval $[n, m]$. Said differently, the left hand side is precisely the full subcategory on filtrations F with associated graded $\mathrm{gr}^k(F)$ only non-trivial for $k \in [n, m]$. Since we restrict to convergent filtrations, this simply means that $F(k) = 0$ for all $k \leq n-1$. We will prove this result by induction on the length $m - n$ of the interval $[n, m]$. For $n = m$, we find that $\mathrm{Fil}(\mathcal{D}(R))^{\heartsuit}_{[n,m]} = \mathcal{D}(R)^{\heartsuit}[n]$, so the claim follows from Proposition 5.27. In the inductive step, let us show that the claim is true for $[n, m+1]$ if it is true for $[n, m]$. We consider the diagram of horizontally written out filtrations

$$\begin{tikzcd}
0 \arrow[r] & F(n) \arrow[r] \arrow[d] & \dots \arrow[r] & F(m) \arrow[r, equal] \arrow[d] & F(m) \arrow[r, equal] \arrow[d] & \dots \\
0 \arrow[r] & F(n) \arrow[r] \arrow[d] & \dots \arrow[r] & F(m) \arrow[r] \arrow[d] & F(m+1) \arrow[r, equal] \arrow[d] & \dots \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{gr}^{m+1}(F) \arrow[r, equal] \arrow[d] & \dots \\
0 \arrow[r] & F(n)[1] \arrow[r] & \dots \arrow[r] & F(m)[1] \arrow[r, equal] & F(m)[1] \arrow[r] & \dots
\end{tikzcd}$$

By the inductive hypothesis, the top row is equivalently given by a chain complex in $\mathrm{Ch}(\mathrm{Mod}(R))_{[n,m]}$. Therefore, it suffices to argue that the "classifying map" going from 3rd to 4th row is described by equivalent data in filtered objects and chain complexes, respectively. In chain complexes, the classifying map is simply a map

$$\mathrm{gr}^{m+1}(F)[-m-1] \to \ker(\mathrm{gr}^{m}(F)[-m] \to \mathrm{gr}^{m-1}(F)[-m+1]).$$

Therefore, we need to see that maps from $\mathrm{gr}^{m+1}(F) \to F(m)[1]$ are also given by maps to this kernel (up to shifts). For this, we recall the fibre sequence

$$F(m)/F(m-2)[1] \to \mathrm{gr}^{m}(F)[1] \to \mathrm{gr}^{m-1}(F)[2]$$

and map from $\mathrm{gr}^{m+1}(F)$ to it. Using that $\mathrm{gr}^{m+1}(F) \in \mathcal{D}(R)_{\geq m+1}$, we find an equivalence

$$\mathrm{Map}(\mathrm{gr}^{m+1}(F), F(m)[1]) \xrightarrow{\simeq} \mathrm{Map}(\mathrm{gr}^{m+1}(F), F(m)/F(m-2)[1])$$

so we obtain a fibre sequence

$$\mathrm{Map}(\mathrm{gr}^{m+1}(F), F(m)[1]) \to \mathrm{Map}(\mathrm{gr}^{m+1}(F), \mathrm{gr}^{m}(F)[1]) \to \mathrm{Map}(\mathrm{gr}^{m+1}(F), \mathrm{gr}^{m-1}(F)[2]).$$

Since the latter two terms are discrete, using again Proposition 5.27, we find that the first term is as claimed.

Now, for $F \in \mathrm{Fil}(\mathcal{D}(R))^{\heartsuit}$, and $n \leq m$, we can consider the following construction: We let $F_{n,m}(k) = 0$ for $k < n$, $F_{n,m}(k) = F(k)$ for $k \in [m, n]$, and $F_{n,m}(k) = F(m)$ for $k > m$. There are then canonical maps

$$F_{-n,n} \to F_{-n-1,n+1} \to \dots$$

whose colimit is given by F. On the level of chain complexes, this constructs a filtration of the chain complex associated to F given by chain complexes whose terms are trivial outside an interval of the form $[-n, n]$. This shows that the functor under investigation is essentially surjective. To see that it is also fully faithful, we then use that filtrations with only finitely many non-trivial terms in the associated graded are compact in $\mathrm{Fil}(\mathcal{D}(R))^{\heartsuit}$. The same is true for the images of these objects in $\mathrm{Ch}(\mathrm{Mod}(R))$, so that fully faithfulness also follows from the above calculation. $\square$

As a consequence, we obtain the following functor.

5.37. **Definition** We define the composite

$$\mathrm{Ch}(\mathrm{Mod}(R)) \simeq \mathrm{Fil}(\mathcal{D}(R))^{\heartsuit} \to \mathcal{D}(R)$$

and think of it as sending a chain complex to its underlying object in the derived category.

5.38. **Remark** The above functor is made to be compatible with homology: The homology of a chain complex identifies canonically with the underlying object of the associated filtered object of $\mathcal{D}(R)$. In particular, the above functor sends quasi-isomorphisms to equivalences, and consequently induces a functor

$$\mathrm{Ch}(\mathrm{Mod}(R))[\mathrm{qiso}^{-1}] \to \mathcal{D}(R).$$

5.39. **Proposition** *The functor* $\mathrm{Ch}(\mathrm{Mod}(R))[\mathrm{qiso}^{-1}] \to \mathcal{D}(R)$ *is an equivalence of* ∞*-categories.*

Exercise. Let $f\colon A \to B$ be a map in $\mathrm{Mod}(R)$. View the sequence

$$0 \to A \to B \to 0$$

as a chain complex in $\mathrm{Mod}(R)$. Work out what the resulting object in $\mathrm{Fil}(\mathcal{D}(R))$ and $\mathcal{D}(R)$.

5.40. **Remark** The functor $\mathrm{Ch}(\mathrm{Mod}(R)) \to \mathcal{D}(R)$ factors through $\mathcal{K}(R)$ which is the localisation of $\mathrm{Ch}(\mathrm{Mod}(R))$ at the homotopy equivalences. One can show that $\mathcal{K}(R)$ is a stable ∞-category. A cofibre of a morphism is represented by the mapping cone construction in chain complexes. The functor $\mathcal{K}(R) \to \mathcal{D}(R)$, being the localisation at the quasi-isomorphisms, can also be described as the Verdier quotient of $\mathcal{K}(R)$ by the stable subcategory given by the acyclic complexes (Exercise: Show that this is a stable subcategory).

All the above tells us that every object in $\mathcal{D}(R)$ is represented by a chain complex, but in a highly non-canonical way. Finding a canonical representative is equivalent to enhancing the object in $\mathcal{D}(R)$ to a filtered object lying in $\mathrm{Fil}(\mathcal{D}(R))^{\heartsuit}$.

5.41. **Remark** For every abelian category $\mathcal{A}$, there exists a derived ∞-category $\mathcal{D}(\mathcal{A})$, which in case $\mathcal{A} = \mathrm{Mod}(R)$ coincides with $\mathcal{D}(R)$ in the way described earlier. This can be constructed similarly as in Remark 5.40 by inverting quasi-isomorphisms in $\mathrm{Ch}(\mathcal{A})$. There is yet another perspective on this provided $\mathcal{A}$ has enough projectives. Namely, one can form the *non-abelian derived* category of $\mathcal{A}$ by taking $\mathrm{PSh}_\Sigma(\mathcal{A}_{\mathrm{proj}})$. This turns out to be a pre-stable ∞-category - that is: It embeds as the connective part of a t-structure in its stabilisation. In general, abelian categories need not have enough projectives, but $\mathrm{Mod}(R)$ for a ring R and $\mathrm{Cond}(\mathrm{Ab})$ do have this. So either of these construction gives a definition of $\mathcal{D}(\mathrm{Cond}(\mathrm{Ab}))$. In the next section, we shall give yet another perspective on the derived category of $\mathrm{Cond}(\mathrm{Ab})$.

5.2. **Sheaves with values in ∞-categories.** Next, we wish to talk about sheaves with values in general ∞-categories, mostly to define and discuss cohomology of condensed abelian groups. Therefore, we will first discuss how to define condensed objects in an ∞-category. We need some definitions. Throughout, our target category $\mathcal{D}$ is assumed to be an ∞-category, bicomplete for convenience.

5.42. **Definition** Let $(\mathcal{C}, \tau)$ be a (small) site. Then we let $\mathrm{Sh}(\mathcal{C}; \mathcal{D})$ be the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) = \mathrm{PSh}(\mathcal{C}; \mathcal{D})$ on objects F, such that for all objects $X \in \mathcal{C}$ and all covering sieves $S \in \mathrm{Cov}_\tau(X)$, the map

$$F(X) \to F(S)$$

is an equivalence.

5.43. **Remark** We explain the notation $F(S)$. Recall that $S \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}) = \mathrm{PSh}(\mathcal{C}; \mathrm{An})$. In particular, $S = \mathrm{colim}_{Y \in \mathcal{C}_{/S}} y(Y)$, and we set

$$F(S) = \lim_{Y \in \mathcal{C}_{/S}} F(Y).$$

In other words, using that $\mathcal{D}$ is complete, we use the equivalence of categories

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{colim}}(\mathrm{PSh}(\mathcal{C}; \mathrm{An}); \mathcal{D}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Fun}^{\lim}(\mathrm{PSh}(\mathcal{C})^{\mathrm{op}}, \mathcal{D})$$

to extend a (contravariant and $\mathcal{D}$-valued) functor on $\mathcal{C}$ to a functor on $\mathrm{PSh}(\mathcal{C}; \mathrm{An})$.

As in the 1-categorical case, we have the following result.

5.44. **Proposition** *The inclusion* $\mathrm{Sh}(\mathcal{C}; \mathcal{D}) \subseteq \mathrm{PSh}(\mathcal{C}; \mathcal{D})$ *admits a left adjoint L. In particular,* $\mathrm{Sh}(\mathcal{C}; \mathcal{D})$ *is presentable if $\mathcal{D}$ is. If finite limits commute with filtered colimits in $\mathcal{D}$, then the sheafification functor L commutes with finite limits.*

5.45. **Remark** Finite limits commute with filtered colimits in $\mathcal{D}$ for instance when $\mathcal{D} = \mathrm{An}$ or when $\mathcal{D}$ is stable. We also note that when $\mathcal{D}$ is stable, then $\mathrm{PSh}(\mathcal{C}; \mathcal{D})$ is also stable, and $\mathrm{Sh}(\mathcal{C}; \mathcal{D}) \subseteq \mathrm{PSh}(\mathcal{C}; \mathcal{D})$ is a full subcategory closed under finite limits, shifts, and filtered colimits and is therefore in particular a stable subcategory.

5.46. **Remark** Let $\mathcal{D}$ be as above. Then there is a canonical equivalence of ∞-categories

$$\mathrm{Fun}^{\lim}(\mathrm{Sh}(\mathcal{C}; \mathrm{An})^{\mathrm{op}}, \mathcal{D}) \to \mathrm{Sh}(\mathcal{C}; \mathcal{D}).$$

In particular, $\mathcal{D}$-valued sheaves may be recovered from the ∞-category of (anima valued) sheaves of $\mathcal{C}$. In fact, if $\mathcal{D}$ is presentable, then the former is a formula for the Lurie tensor product $\mathrm{Sh}(\mathcal{C}; \mathrm{An}) \otimes \mathcal{D}$ between the two presentable ∞-categories $\mathrm{Sh}(\mathcal{C}; \mathrm{An})$ and $\mathcal{D}$. We think of this as an analog of the fact that $C(X) \otimes A \cong C(X; A)$ for $X \in \mathrm{CH}$ and continuous $\mathbb{C}$-valued functions $C(X)$ on X, a further C^*-algebra A and any of your favourite C^*-tensor product (this in particular says that all such tensor products agree).

More generally, we have

$$\mathrm{Sh}(\mathcal{C}; \mathcal{D}') \otimes \mathcal{D} \simeq \mathrm{Sh}(\mathcal{C}; \mathcal{D}' \otimes \mathcal{D}).$$

This is indeed a generalisation of the previous equivalence since An is the tensor unit of the symmetric monoidal category of presentable ∞-categories (with left adjoint functors as morphisms) for the Lurie tensor product.

A $\mathcal{D}$-valued sheaf on $\mathcal{C}$ can satisfy a stronger property called being *hypercomplete* - there is an analog of this also in the case also where $\mathcal{D}$ is an ordinary category, but then any sheaf is automatically hypercomplete (ordinary sheaves satisfy hyperdescent, not only descent). It will turn out that condensed objects in $\mathcal{D}$ should be defined not as $\mathcal{D}$-valued sheaves on CH_κ, but rather as $\mathcal{D}$-valued hypercomplete sheaves on CH_κ. Let us therefore introduce the relevant definitions to define what a hypercomplete sheaf is.

5.47. **Definition** Let $\mathcal{X}$ be a cocomplete ∞-category and $X \in \mathcal{X}$ an object. For a map $f\colon Y \to X$, we define its Cech-nerve $\check{C}_\bullet(f)$ as the simplicial object obtained by right Kan extension in the diagram

$$\begin{tikzcd} \Delta^{\mathrm{op}}_{\leq 0} \arrow[r, "f"] \arrow[d] & \mathcal{X}_{/X} \\ \Delta^{\mathrm{op}} \arrow[ur, dashed] & \end{tikzcd}$$

We say that the morphism f is descent-effective, if the canonical map $\operatorname*{colim}_{\Delta^{\mathrm{op}}} \check{C}_\bullet(f) \to X$ is an equivalence.

5.48. **Example** Let $\mathcal{X}$ above be $\mathrm{Sh}(\mathcal{C})$ for a site $\mathcal{C}$ and let $X \in \mathcal{C} \subseteq \mathcal{X}$. Let $S \in \mathrm{Cov}_\tau(X)$ be a τ-covering of X generated by a covering family $\{X_i \to X\}_{i\in I}$. Then we may consider $f\colon Y = \amalg_{i\in I} y(X_i) \to y(X)$. We have argued earlier that the canonical map

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}} \check{C}_\bullet(f) \to S$$

is an equivalence in $\mathrm{PSh}(\mathcal{C})$. Furthermore, after sheafification, the map $S \to X$ becomes an equivalence (by definition of sheaves). In total we obtain that the canonical map

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}} \check{C}_\bullet(f) \to X$$

is an equivalence in $\mathrm{Sh}(\mathcal{C})$, so that f is descent-effective. In other words, coverings of τ always give rise to descent-effective morphisms in sheaves, and in fact, the descent-effective morphisms in sheaves conversely determine the topology.

5.49. **Definition** Let $X\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object. For $n \geq 0$, we define skeleton and coskeleton functors sk_n and cosk_n by the formulas

$$\mathrm{sk}_n = (\iota_n)_!(\iota_n)^* \quad \text{and} \quad \mathrm{cosk}_n = (\iota_n)_*(\iota_n)^*$$

where $\iota_n\colon \Delta^{\mathrm{op}}_{\leq n} \to \Delta^{\mathrm{op}}$ is the inclusion. We use the same definition for semi-simplicial objects. A (semi)-simplicial object X, where the canonical map $X \to \mathrm{cosk}_n(X)$ is an equivalence is called n-coskeletal.

5.50. **Definition** Let $\mathcal{X}$ be a bicomplete ∞-category and $X \in \mathcal{X}$ an object. A (semi)-simplicial object $U\colon \Delta^{\mathrm{op}}_{\mathrm{inj}} \to \mathcal{X}_{/X}$ is called a hypercover of X if the canonical map

$$U_n \to (\mathrm{cosk}_{n-1}U)_n$$

is descent-effective for every $n \geq 0$.

5.51. **Remark** A hypercover U of X satisfies that $U_0 \to X$ is descent-effective, that $U_1 \to U_0 \times_X U_0$ is descent-effective and so on.

5.52. **Example** Let $f\colon Y \to X$ be a descent-effective morphism in $\mathcal{X}$. Then $\check{C}_\bullet(f)$ is a hypercover of X. Indeed, for $n = 0$, the map to investigate is descent effective by assumption and for $n \geq 1$, it is an equivalence, so in particular descent-effective.

5.53. **Definition** Let $(\mathcal{C}, \tau)$ be a site and $\mathcal{D}$ an ∞-category. A sheaf $F \in \mathrm{Sh}(\mathcal{C}; \mathcal{D})$ is a hypersheaf if for all objects $X \in \mathrm{Sh}(\mathcal{C}, \mathcal{D})$ and all hypercovers U of X, the canonical map

$$F(X) \to F(U) = \lim_{[n]\in\Delta} F(U_n)$$

is an equivalence. We write $\mathrm{Sh}^{\mathrm{hyp}}(\mathcal{C}; \mathcal{D}) \subseteq \mathrm{Sh}(\mathcal{C}; \mathcal{D})$ for the full subcategory of hypercomplete sheaves.

Exercise. Let F be a sheaf of sets on a site $\mathcal{C}$ (or more generally a sheaf with values in any 1-category). Show that F is hypercomplete.

5.54. **Remark** In [**?**, A.5.3.3], it is shown that when U is a semi-simplicial hypercover of X, then the canonical map

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}} U_n \to X$$

is an equivalence in $\mathrm{Sh}^{\mathrm{hyp}}(\mathcal{C}; \mathcal{D})$. In addition, a hypercover is also a semi-simplicial hypercover, so in the above definition, we are also allowed to replace hypercover by semi-simplicial hypercover. It is technically convenient to sometimes only construct semi-simplicial hypercovers, but then we find that hypercomplete sheaves satisfy descent also with respect to these semi-simplicial hypercovers. See the proof of Theorem 5.57 for an example where it is more convenient to construct semi-simplicial hypercovers than actual hypercovers.

5.55. **Proposition** *The inclusion* $\mathrm{Sh}^{\mathrm{hyp}}(\mathcal{C}; \mathcal{D}) \subseteq \mathrm{Sh}(\mathcal{C}; \mathcal{D})$ *admits a left adjoint. In particular, hypercomplete sheaves form a full subcategory of all sheaves closed under limits.*

5.56. **Corollary** *In case* $\mathcal{D}$ *is stable, we find that* $\mathrm{Sh}^{\mathrm{hyp}}(\mathcal{C}; \mathcal{D}) \subseteq \mathrm{Sh}(\mathcal{C}; \mathcal{D}) \subseteq \mathrm{PSh}(\mathcal{C}; \mathcal{D})$ *are each full stable subcategories.*

Proof. In remains only to note that hypercomplete sheaves are closed under shifting (which is always performed pointwise in $\mathcal{D}$). ☐

For the sites we have studied in this course, we then obtain the following results:

5.57. **Theorem** *Let* κ *be a strong limit cardinal and* $\mathcal{D}$ *a complete* ∞*-category. The restriction functors induce equivalences*

$$\mathrm{Sh}(\mathrm{CH}_\kappa; \mathcal{D}) \xrightarrow{\simeq} \mathrm{Sh}(\mathrm{tdCH}_\kappa; \mathcal{D})$$

as well as

$$\mathrm{Sh}^{\mathrm{hyp}}(\mathrm{CH}_\kappa; \mathcal{D}) \xrightarrow{\simeq} \mathrm{Sh}^{\mathrm{hyp}}(\mathrm{tdCH}_\kappa; \mathcal{D}) \xrightarrow{\simeq} \mathrm{Sh}^{\mathrm{hyp}}(\mathrm{edCH}_\kappa; \mathcal{D}).$$

Proof. All equivalences not involving edCH_κ are proven in a similar vein in Theorem 4.5. To involve extremally disconnected spaces, we need the following general result: Let $X \in \mathrm{CH}_\kappa$. Then there exists a simplicial object in $(\mathrm{edCH}_\kappa)_{/X}$ which is a hypercover when viewed as a simplicial object in $\mathrm{Sh}(\mathrm{CH}_\kappa) = \mathrm{Cond}_\kappa(\mathrm{Set})$. In fact, using Remark 5.54 it suffices to construct a semi-simplicial hypercover. Once this is true, the argument of Theorem 4.5 goes through: One shows that the right Kan-extension functors on the level of presheaves restricts to hypercomplete sheaves and is therefore fully faithful. In all cases above, one shows that the restriction functors are conservative, so that the restriction and right Kan extension adjunction induces the desired (adjoint) equivalence.

It remains to explain something about finding a hypercover consisting of extremally disconnected spaces of $X \in \mathrm{CH}_\kappa$. For this, we argue inductively. Namely, we first choose a surjection $f\colon Y \to X$ with $Y \in \mathrm{edCH}_\kappa$ and form the Cech-nerve $\check{C}_\bullet(f)$. Now, inductively assume that we have constructed a semi-simplicial object U such that for $k \leq n$, the map $U_k \to (\mathrm{cosk}_{k-1} U)_n$ is descent-effective and $U_k \in \mathrm{edCH}_\kappa$. Pick a surjection $Y' \to U_{n+1}$ with $Y' \in \mathrm{edCH}_\kappa$. By composition, we obtain a diagram $U'\colon \Delta^{\mathrm{op}}_{\leq n+1}$ with a map to $\iota^*_{n+1} U$ such that $U'_k \to U_k$ is an isomorphism for $k \leq n$. □

5.58. **Definition** Let $\mathcal{C}$ be bicomplete ∞-category. We let $\mathrm{Cond}_\kappa(\mathcal{C})$ denote the ∞-category of hypercomplete $\mathcal{C}$-valued sheaves on CH_κ, tdCH_κ, or edCH_κ.

Similarly to the case of κ-condensed sets, we have the following results.

5.59. **Lemma** *The canonical functors*

$$\mathrm{Cond}_\kappa(\mathcal{C}) \to \mathrm{Sh}(\mathrm{edCH}_\kappa; \mathcal{C}) \to \mathrm{Fun}^\times(\mathrm{edCH}_\kappa^{\mathrm{op}}, \mathcal{C})$$

are equivalences.

Proof. The latter equivalence is similar as in the 1-categorical case: It suffices to show that a product preserving functor also satisfies descent for type (3) families, i.e. those consisting of a single surjection. But the sieve generated by such a surjection $f\colon Y \to X$ is equal to $y(X)$, so descent is automatic. It then remains to argue that any sheaf on edCH_κ is hypercomplete. □

5.60. **Corollary** *For $\kappa' \geq \kappa$, the left Kan extension functor*

$$\mathrm{Cond}_\kappa(\mathcal{C}) \to \mathrm{Cond}_{\kappa'}(\mathcal{C})$$

is fully faithful and commutes with λ-small limits if $\mathcal{C} = \mathrm{An}$.

Just as in the case of condensed sets, we have the following definition.

5.61. **Definition** We then define $\mathrm{Cond}(\mathcal{C}) = \bigcup_\kappa \mathrm{Cond}_\kappa(\mathcal{C})$.

5.62. **Remark** The characterisation of Proposition 4.32 for which objects X of $\mathrm{Fun}^\times(\mathrm{edCH}^{\mathrm{op}}, \mathcal{C})$ belong to $\mathrm{Cond}(\mathcal{C})$ remains true: It is so if and only if there exists a regular cardinal κ such that for all κ-cofiltered limits $S = \lim_i S_i$ of totally disconnected spaces, we have that the map

$$\operatorname*{colim}_i X(S_i) \to X(S)$$

is an equivalence.

5.63. **Proposition** *Suppose $\mathcal{C}$ has a closed symmetric monoidal structure. Then* $\mathrm{Cond}(\mathcal{C})$ *has an induced closed symmetric monoidal structure. The formula for the tensor product and internal hom are the same as in Proposition 5.5.*

Cohomology and Ext. We now consider the ∞-category $\mathrm{Cond}(\mathcal{D}(\mathbb{Z}))$ of condensed objects in the derived category of $\mathbb{Z}$. By what we have just argued, there is a canonical fully faithful functor

$$\mathrm{Cond}(\mathrm{Ab}) \to \mathrm{Cond}(\mathcal{D}(\mathbb{Z})).$$

5.64. **Definition** Let $A, B \in \mathrm{Cond}(\mathrm{Ab})$ be condensed abelian groups. We define Ext groups between them as follows

$$\mathrm{Ext}^n_{\mathrm{Cond}}(A, B) = \pi_{-n}\mathrm{map}_{\mathrm{Cond}(\mathcal{D}(\mathbb{Z}))}(A, B)$$

We find that $\mathrm{Ext}^0(A, B) = \mathrm{Hom}_{\mathrm{Cond(Ab)}}(A, B)$ is simply the abelian group of homomorphisms from A to B inside condensed abelian groups, just as we are used to the fact that $\mathrm{Ext}^0_R(M, N) = \mathrm{Hom}_R(M, N)$ for R-modules.

We continue with a word about how we can calculate Ext-groups by means of projective resolutions. For this, let $A \in \mathrm{Cond}(\mathrm{Ab})$. One can find a hypercover U of A with U_n of the form $\bigoplus \mathbb{Z}[T_i]$ and $T_i \in \mathrm{edCH}$. We then find that $|U_\bullet| \simeq A$ in $\mathrm{Cond}(\mathcal{D}(\mathbb{Z}))$. Consequently, upon mapping to B, we obtain an equivalence

$$\mathrm{map}(A, B) \simeq \lim_{\Delta} \prod B(T_i)$$

where the $B(T_i)$ is an abelian group. In fact, this makes the diagram over which we take a limit a cosimplicial abelian group - and the Dold-Kan correspondence tells us that we may equivalently think of it as a chain complex concentrated in non-positive degrees. Even more concretely, the differentials are given by the alternating sum over the maps appearing in the cosimplicial abelian group. We use here crucially the fact that the $\mathbb{Z}[T_i]$ are projective and hence the spectrum of maps $\mathrm{map}(\mathbb{Z}[T_i], B)$ is really discrete.

6. Collection of exercises

6.1. Category theory.

Exercise. Show that a natural transformation $\tau\colon F \to G$ is a natural isomorphism if and only if for all objects X, the maps $F(X) \to G(X)$ are isomorphisms.

Exercise. Show that a colimit over the functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ is a terminal object.

Exercise. Is the forgetful functor $\mathrm{Top} \to \mathrm{Set}$ conservative?

Exercise. Prove or disprove that the inclusion $\mathrm{CH} \to \mathrm{Top}$ admits a right adjoint.

Exercise. Prove Lemma 3.2 in the case $k = 1$ directly, i.e. show that the displayed maps are isomorphisms.

Exercise. Calculate $\lim_{\Delta^{\mathrm{op}}} X$ and $\mathrm{colim}_{\Delta} Y$.

Exercise. Calculate the colimit of the functor $y\colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$.

Exercise. Let $F\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon G$ be an adjunction. Suppose given full subcategories $\mathcal{C}_0 \subseteq \mathcal{C}$ and $\mathcal{D}_0 \subseteq \mathcal{D}$ so that the functors F and G restrict to functors on these subcategories. Then, these functors again form an adjunction $F\colon \mathcal{C}_0 \leftrightarrows \mathcal{D}_0\colon G$.

Exercise. Let $\mathcal{C}$ be a small category and let F be a presheaf on $\mathcal{C}$. Show that the canonical map

$$\operatorname*{colim}_{X \to F} y(X) \to F$$

is an isomorphism.

Proof. We use the Yoneda lemma and calculate maps from each term to another presheaf G. On the left we have

$$\lim_{X \to F} G(X)$$

and need to see that this is canonically equivalent to the set of natural transformations between F and G. Both are subsets of the set

$$\prod_{X \in \mathcal{C}} \operatorname{Hom}(F(X), G(X)) \cong \prod_{X \in \mathcal{C}} \prod_{F(X)} G(X) \cong \prod_{X \to F} G(X)$$

and one checks that these subsets correspond to each other. $\square$

The content of the following exercise is often phrased as "colimits in $\operatorname{PSh}(\mathcal{C})$ are universal".

Exercise. Let $\mathcal{C}$ be a small category and let $\operatorname{PSh}(\mathcal{C})$ be its category of presheaves. Let $F \to G$ be a morphism of presheaves and suppose given an isomorphism $\operatorname*{colim}_{i \in I} G_i \cong G$. Set $F_i = F \times_G G_i$. Then show that the canonical map $\operatorname*{colim}_{i \in I} F_i \to F$ is an isomorphism.

Proof. $\square$

While we are at it, we will also need the following result.

Exercise. Let $f \colon \mathcal{C} \to \mathcal{D}$ be a functor between (essentially) small categories. Let $\mathcal{E}$ be a (co)complete category. Then the functor $f^* \colon \operatorname{Fun}(\mathcal{D}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{E})$ has a left/right adjoint denoted by $f_!$ and f^*, respectively. These are the left and right Kan extensions along f. For $F \in \operatorname{Fun}(\mathcal{C}, \mathcal{E})$, they are given by the following formulas

$$f_!(F)(d) = \operatorname*{colim}_{c \in \mathcal{C}_{/d}} F(c) \quad \text{and} \quad f_*(F)(d) = \lim_{c \in \mathcal{C}_{d/}} F(c).$$

Here, the slice categories appearing in the formulas are given by the following two pullbacks:

$$\begin{tikzcd} \mathcal{C}_{/d} \arrow[r] \arrow[d] & \mathcal{D}_{/d} \arrow[d] \\ \mathcal{C} \arrow[r, "f"] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}_{d/} \arrow[r] \arrow[d] & \mathcal{D}_{d/} \arrow[d] \\ \mathcal{C} \arrow[r, "f"] & \mathcal{D} \end{tikzcd}$$

Proof. We treat the formula for left Kan extensions first. The universal property of presheaves gives a colimit preserving functor $f_! \colon \operatorname{PSh}(\mathcal{C}) \to \operatorname{PSh}(\mathcal{D})$ which fits inside a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d] & \mathcal{D} \arrow[d] \\ \operatorname{PSh}(\mathcal{C}) \arrow[r, "f_!"] & \operatorname{PSh}(\mathcal{D}) \end{tikzcd}$$

in which the vertical functors are the Yoneda embeddings. Let $\mathcal{E}$ be cocomplete and consider then the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{D},\mathcal{E}) \arrow[r, "f^*"] & \mathrm{Fun}(\mathcal{C},\mathcal{E}) \\
\mathrm{Fun}^{\mathrm{colim}}(\mathrm{PSh}(\mathcal{D}),\mathcal{E}) \arrow[u] \arrow[r, "(f_!)^*"] & \mathrm{Fun}^{\mathrm{colim}}(\mathrm{PSh}(\mathcal{C}),\mathcal{E}) \arrow[u]
\end{tikzcd}$$

where the super script refers to the full subcategory of all functors which preserve colimits. Note that we use here that $f_!$ preserves colimits. The diagram then commutes because it essentially arises from the above diagram by applying $\mathrm{Fun}(-,\mathcal{E})$. The universal property of presheaves also gives that the vertical maps in this diagram are equivalences. Therefore, in order to calculate the left adjoint of the upper horizontal functor f^*, we may equivalently calculate the right adjoint of the lower horizontal functor. For this, we note the adjunction

$$f_! \colon \mathrm{PSh}(\mathcal{C}) \leftrightarrows \mathrm{PSh}(\mathcal{D}) \colon f^*$$

and note that both functors preserve colimits (since both functors are left adjoints). Therefore, we obtain an adjunction

$$(f^*)^* \colon \mathrm{Fun}^{\mathrm{colim}}(\mathrm{PSh}(\mathcal{C}),\mathcal{E}) \leftrightarrows \mathrm{Fun}^{\mathrm{colim}}(\mathrm{PSh}(\mathcal{D}),\mathcal{E}) \colon (f_!)^*$$

where now $(f_!)^*$ is the right adjoint. Therefore, we are reduced to calculating the values of the functor $(f^*)^*$. For this, let $F \in \mathrm{Fun}(\mathcal{C},\mathcal{E})$, viewed as a colimit preserving functor $\mathrm{PSh}(\mathcal{C}) \to \mathcal{E}$. Then $(f^*)^*(F)$ is given by the composite

$$\mathcal{D} \xrightarrow{y} \mathrm{PSh}(\mathcal{D}) \xrightarrow{f^*} \mathrm{PSh}(\mathcal{C}) \xrightarrow{F} \mathcal{E}.$$

Evaluating this on an object $d \in \mathcal{D}$, we first note that $f^*(y(d)) = \mathrm{Map}_{\mathcal{C}}(f(-),d)$. In order to calculate F is this presheaf, we need to write it as a colimit over representables. This is given by the following. We consider the functor

$$\mathcal{C}_{/d} \to \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$$

and note that its colimit is given precisely by $\mathrm{Map}_{\mathcal{C}}(f(-),d)$. Consequently, we deduce

$$(f^*)^*(F)(d) = \operatorname*{colim}_{c\in\mathcal{C}_{/d}} F(c)$$

claimed. $\square$

6.2. Point-set topology.

Exercise. Show that connected components of a topological space are open if X is locally connected.

Exercise. Recall the notion of path connected spaces. Discuss the relation between (locally) path connected and (locally) connected spaces.

Exercise. Let X be a topological space and $A \subseteq X$ a subset. Show that a point x lies in $\overline{A}$ if and only if there is a Filter $\mathcal{F}$ on A which converges to x in the sense that $\mathcal{U}(x) \cap A \subseteq \mathcal{F}$.

Exercise. Show the following assertions.

(1) Subspaces of totally disconnected spaces are totally disconnected.
(2) Products of Hausdorff spaces are Hausdorff.

(3) Let $f\colon X \to Y$ with Y Hausdorff be a map of sets between topological spaces. If f is continuous, then $\Gamma_f \subseteq X \times Y$ is closed. If Y is compact Hausdorff and $\Gamma_f \subseteq X \times Y$ is closed, then f is continuous. Show that these assumptions are necessary.

Exercise. Prove or disprove the following statements:

(1) A subspace of an extremally disconnected space is extremally disconnected.
(2) An open subspace of an extremally disconnected space is extremally disconnected.
(3) A closed subspace of an extremally disconnected space is extremally disconnected.
(4) A dense subspace of an extremally disconnected space is extremally disconnected.

Proof. That (1) is wrong follows from the proof of (3). For (3) consider any discrete set M. Then $M \to \beta(M)$ is injective and in fact a homeomorphism onto the image. Therefore $\beta(M)\setminus M$ is a closed subspace of an extremally disconnected compact Hausdorff space. But in general, it is not again extremally disconnected. For this, it suffices to construct to open and disjoint sets in $\beta(M) \setminus M$ whose closures are not disjoint anymore. For definiteness consider $M = \mathbb{N}$, the natural numbers. The basic opens of the topology on $\beta(M)$ are the sets $[A]$ for $A \subseteq \mathbb{N}$.

(2) and (4) are true. $\square$

Exercise. Show that extremally disconnected compact Hausdorff spaces are *projective* in the category CH of compact Hausdorff spaces, that is for each solid diagram, the dashed arrow as indicated in the below diagram exists, rendering the diagram commutative:

$$\begin{tikzcd} & Z \arrow[d, two heads, "f"] \\ X \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Here, X is extremally disconnected compact Hausdorff, Y and Z are compact Hausdorff, and f is surjective.

Exercise. Let X be a compact Hausdorff space. Show that the $\mathbb{C}$-vector space $C(X)$ of continuous $\mathbb{C}$-valued functions is finite dimensional if and only if X is finite.

Exercise. Let X be an infinite compact metric space. Show that there exists a continuous function $f\colon X \to \mathbb{R}$ with infinite image.

Proof. Suppose that every continuous map $f\colon X \to \mathbb{R}$ has finite image. Consider $x \in X$ and the map $f = d(x, -)\colon X \to \mathbb{R}$. Then $f^{-1}(0) = \{x\}$ by the axioms of a metric. Since f has finite image, we conclude that $\{x\}$ is open and therefore that X is discrete. Since it also compact, it must be finite. $\square$

Exercise. With the notation of the proof of Theorem 2.57, show that the association $A \mapsto [A]$ has the following properties.

(1) $[\varnothing] = \varnothing$ and $[M] = \beta(M)$.
(2) $[A] \subseteq [B] \Leftrightarrow A \subseteq B$,
(3) $[A] = [B] \Leftrightarrow A = B$,
(4) $[A] \cup [B] = [A \cup B]$
(5) $[A] \cap [B] = [A \cap B]$

(6) $[M \setminus A] = \beta(M) \setminus [A]$.

Exercise. Let $A \subseteq M$ be a subset. Then show that $\overline{A} \subseteq \beta(M)$ coincides with $[A]$.

Exercise. Given what we have shown, prove that there is at most one continuous extension of $f\colon M \to Y$ to $\beta(M)$.

Exercise. Let X be a compact Hausdorff space. Consider an open set U which contains x. Show that there is an open set V also containing x and such that $\overline{V} \subseteq U$.

Exercise. Let X be a set and $\mathcal{P}(X)$ its power set, i.e. the set of subsets of X.

(1) Show that there is a bijection $\mathcal{P}(X) \to \{0,1\}^X$, given by sending $Y \subseteq X$ to the function
$$x \mapsto \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}.$$
(2) Show that the set of ultrafilters on X is a subset of $\mathcal{P}(\mathcal{P}(X))$, and therefore of $\{0,1\}^{\mathcal{P}(X)}$. Deduce that $|\beta(M)| < 2^{2^{|M|}}$.
(3) Equip $\{0,1\}^{\mathcal{P}(X)}$ with the product topology. Show that $\beta(X^\delta)$ has the subspace topology of the product topology. Show that $\beta(X^\delta)$ is a closed subset. Deduce that $\beta(X^\delta)$ is totally disconnected, compact Hausdorff.

Exercise. Show that every extremally disconnected compact Hausdorff space is a retract of the Stone-Cech compactification of a discrete set.

6.3. Sheaves and Sites.

Exercise. Show that in the notation of the proof of Theorem 3.14, the topologies τ and τ' agree.

Exercise. Let $p\colon X' \to X$ be a map with a section $s\colon X \to X'$ in some category $\mathcal{C}$ which admits pullbacks. Let F be a presheaf on $\mathcal{C}$. Show (by hand) that the canonical diagram
$$F(X) \to F(X') \rightrightarrows F(X' \times_X X')$$
is an equaliser diagram.

Exercise. Show that a map $f\colon X \to Y$ in a category is an epimorphism if and only if the diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[d, "\mathrm{id}"] \\ Y \arrow[r, "\mathrm{id}"] & Y \end{tikzcd}$$
is a pushout.

Exercise. Show that a map of sheaves is an isomorphism if and only if it is a monomorphism and an epimorphism.

Proof. The proof is from Prop. 4.2 in SGA4.1. It is clear that isomorphisms are monos and epis. So let $F \to G$ be a mono and an epi. Consider the pushout of presheaves

$$\begin{array}{ccc} F & \longrightarrow & G \\ \big\downarrow & & \big\downarrow \\ G & \longrightarrow & K \end{array}$$

and observe that it is a pullback: Indeed, this is so if and only if it is so objectwise, so we need to observe that a pushout of sets

$$\begin{array}{ccc} F(T) & \xrightarrow{u} & G(T) \\ \big\downarrow{\scriptstyle u} & & \big\downarrow \\ G(T) & \longrightarrow & K(T) \end{array}$$

where u is injective is also a pullback. Applying sheafification, we obtain the square

$$\begin{array}{ccc} F & \longrightarrow & G \\ \big\downarrow & & \big\downarrow \\ G & \longrightarrow & L(K) \end{array}$$

which is again a pushout and a pullback in sheaves. Since $F \to G$ is epi, we deduce from the previous exercise that the map $G \to L(K)$ is an isomorphism, and consequently that $F \to G$ is also an isomorphism (being the pullback of an isomorphism). □

Exercise. A map $f: F \to G$ of sheaves is an epimorphism if and only if for all object c in the site and all maps $c \to G$, the induced map $y(c) \times_G F \to y(c)$ is a covering sieve.

Proof. Prop. 5.1 in SGA4.1. □

Exercise. Show that a map of sheaves is surjective if and only if it is an epimorphism.

References

[Man22] L. Mann, *A p-adic 6-functor formalism in rigid-analytic geometry*, arXiv:2206.02022 (2022).

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