ALMOST COMPLEX 4-MANIFOLDS

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I do not claim any kind of originality for this. The goal is to prove the following

Theorem 1. *Let X be a closed oriented smooth 4-manifold. If X is almost complex then*

$$\sigma(X) + \chi(X) = 0 \mod 4.$$

If X satisfies $b_1(X) = 0$, e.g. if X is simply connected, then the converse is also true.

We start out with basic obstruction theory: We intend to solve the lifting problem desribed in the diagram

$$\begin{array}{ccc} & & BU(2) \\ & \nearrow & \downarrow \\ X & \xrightarrow[\tau_X]{} & BSO(4) \end{array}$$

To understand the homotopy fiber of the map $BU(2) \to BSO(4)$ which is $SO(4)/U(2)$ we observe that it is a manifold of dimension 2 and the long exact sequence in homotopy groups says it is simply connected, thus we have $SO(4)/U(2) \cong S^2$.

This says that we have obstructions to lifting the map $\tau_X \colon X \to BSO(4)$ that lie in $H^3(X; \pi_2(S^2))$ and $H^4(X; \pi_3(S^2))$.

Proposition 2. *The first obstruction is given by the class*

$$\beta(w_2(X)) \in H^3(X; \mathbb{Z}).$$

Proof. The first obstruction is given by applying Postnikov sections to the spaces $BU(2)$ and $BSO(4)$: we have

(1) $\pi_1(BU(2)) = 0 = \pi_1(BSO(4))$, and
(2) $\pi_2(BU(2)) \cong \mathbb{Z}$ and $\pi_2(BSO(4)) \cong \mathbb{Z}/2$ and the map is the projection map.

We thus find that the first obstruction is given by the obstruction to lifting the map

$$\begin{array}{ccccc} & & & & P_2(BU(2)) \simeq K(\mathbb{Z}, 2) \\ & & & \nearrow & \downarrow \\ X & \longrightarrow & BSO(4) & \longrightarrow & P_2(BSO(4)) \simeq K(\mathbb{Z}/2, 2) \end{array}$$

and this obstruction is clearly the Bockstein of $w_2(X)$ because the map $BSO(4) \to K(\mathbb{Z}/2, 2)$ is w_2. ☐

Proposition 3. *For any smooth oriented 4-manifold X, the class $\beta(w_2(X)) \in H^3(X; \mathbb{Z})$ vanishes.*

Proof. The assertion is equivalent to finding an integral lift of $w_2(X)$. So we consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_1(X;\mathbb{Z}), \mathbb{Z}) & \longrightarrow & H^2(X;\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(H_2(X;\mathbb{Z}), \mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ext}(H_1(X;\mathbb{Z}), \mathbb{Z}/2) & \longrightarrow & H^2(X;\mathbb{Z}/2) & \longrightarrow & \mathrm{Hom}(H_2(X;\mathbb{Z}), \mathbb{Z}/2) & \longrightarrow & 0 \end{array}$$

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Now we calculate the image of $w_2(X)$ in $\mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}/2)$. So let $y \in H_2(X;\mathbb{Z})$. Then

$$\begin{aligned} w_2(X)(y) &= \langle w_2(X), \mathrm{red}_2(y)\rangle \\ &= \langle w_2(X) \cup \mathrm{PDred}_2(y), [X]\rangle \\ &= \mathrm{red}_2(\langle \mathrm{PD}(y)^2, [X]\rangle) \end{aligned}$$

Hence the image of $w_2(X)$ lifts to the element in $\mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z})$ given by $x \mapsto \langle \mathrm{PD}(x)^2, [X]\rangle$.

Furthermore the left vertical map is onto because its cokernel is given by $\mathrm{Ext}^2(H_1(X;\mathbb{Z}),\mathbb{Z})$ which is zero since $\mathbb{Z}$ has global dimension one. This implies that there exists a lift of $w_2(X)$ as claimed. □

To interpret the second obstruction we need the following preliminary lemmas.

Lemma 4. *A complex rank 2 bundle over a closed 4-dimensional manifold is uniquely determined by its chern classes. Moreover every choice of first and second chern class is realized by a rank 2 vector bundle.*

Proof. The lemma follows if we show that

$$P_4(BU(2)) \xrightarrow{(c_1,c_2)} K(\mathbb{Z},2)\times K(\mathbb{Z},4)$$

is an equivalence. For this we calculate the homotopy groups of $BU(2)$ which are:

$$\pi_i(BU(2)) \cong \begin{cases} 0 & \text{for } i = 0,1,3 \\ \mathbb{Z} & \text{for } i = 2,4 \end{cases}$$

Thus the k-invariant of $P_4(BU(2))$ lies in $H^5(K(\mathbb{Z},2);\mathbb{Z}) = 0$ so the space is a product of the two Eilenberg MacLane spaces. To see that $c_2\colon \pi_2(BU(2)) \to \mathbb{Z}$ is an isomorphism we notice that $c_2(\gamma_{\mathbb{H}P^1}) = e(\gamma_{\mathbb{H}P^1}) = 1$. □

Lemma 5. *A real oriented rank 4 bundle E over a 4-manifold X is uniquely determined by $p_1(E)$, $w_2(E)$ and $e(E)$.*

Proof. First we show that 2 stable bundles over X are isomorphic if and only if p_1 and w_2 agree. Then we use that a rank 4 bundle which is stably trivial and has euler class $e(E) = 0$ is trivial. For the first part we calculate the homotopy groups to be

$$\pi_i(BSO(4)) \cong \begin{cases} 0 & \text{for } i = 0,1,3 \\ \mathbb{Z}/2 & \text{for } i = 2 \\ \mathbb{Z} & \text{for } i = 4 \end{cases}$$

thus we obtain a fibration sequence

$$K(\mathbb{Z},4) \longrightarrow P_4(BSO(4)) \xrightarrow{w_2} K(\mathbb{Z}/2,2)$$

Moreover the map

$$[X, BSO] \xrightarrow{(w_2,p_1)} H^2(X;\mathbb{Z}/2)\times H^4(X;\mathbb{Z})$$

is a group homomorphism. We now claim that if it is injective. To see this we consider the map $X \xrightarrow{c} S^4$ of degree one. We obtain a commutative diagram

$$\begin{array}{ccccc} H^4(S^4;\mathbb{Z}) & \xrightarrow{\cong} & [S^4, BSO] & \longrightarrow & 0 = H^2(S^4;\mathbb{Z}/2) \\ \downarrow\cong & & \downarrow & & \downarrow \\ H^4(X;\mathbb{Z}) & \longrightarrow & [X, BSO] & \longrightarrow & H^2(X;\mathbb{Z}/2) \end{array}$$

So taking a bundle E over X with $w_2(E) = 0$ it lifts to $H^4(X;\mathbb{Z}) \cong H^4(S^4;\mathbb{Z})$. But now we calculate that

$$p_1(c^*(E)) = c^*(p_1(E)) = 0$$

since we assumed that $p_1(E) = 0$ as well. Moreover $p_1 \colon \pi_4(BSO) \to \mathbb{Z}$ is injective. Thus E is trivial in $[X, BSO]$.

Next we consider the fibration sequence

$$S^n \xrightarrow{\tau_{S^n}} BSO(n) \to BSO(n+1)$$

We want to study maps from an n-manifold M to $BSO(n)$ that become trivial when mapped to $BSO(n+1)$. Because of the exact sequence

$$[M, S^n] \longrightarrow [M, BSO(n)] \longrightarrow [M, BSO(n+1)]$$

we obtain that such bundles lift to an element of $[M, S^n] \cong \mathbb{Z}$. We consider the diagram

$$\begin{array}{ccccc} [M, S^n] & \longrightarrow & [M, BSO(n)] & \longrightarrow & [M, BSO(n+1)] \\ & \searrow & \downarrow{\scriptstyle e} & & \\ & & K(\mathbb{Z}, n) & & \end{array}$$

and claim that the dotted composite is injective. Again by comparing via the degree one map $M \to S^n$ it suffices to show this for $M = S^n$. There the identity of $[S^n, S^n]$ is mapped to the tangent bundle of S^n which is mapped to its euler class, which is 2 if the dimension n is even. Since 4 is even we are done. □

We are now in position to prove the following

Proposition 6. *Let X be an oriented smooth closed 4-manifold. Then X admits an almost complex structure if and only if there exists an element $c \in H^2(X; \mathbb{Z})$ such that*

(1) $\mathrm{red}_2(c) = w_2(X)$, *and*
(2) $c^2 = 3\sigma(X) + 2\chi(X)$.

Proof. If X is almost complex, it is an easy application of the calculation $p_1(\tau_X) = c_1(\tau_X)^2 - 2c_2(\tau_X)$ to see that $c_1(\tau_X)$ is a c as needed, using that $p_1(\tau_X) = 3\sigma(X)$ and that $c_2(\tau_X) = e(\tau_X) = \chi(X)$.

Moreover given such a c, by the previous lemma we find a complex bundle $E \in [X, BU(2)]$ with $c_2(E) = e(\tau_X)$ and $c_1(E) = c$. Applying the second lemma and the equations of the assumptions implies that the underlying real bundle of E is τ_X as needed. □

We are now in ready to prove the main theorem.

First we show that if X is almost complex, then $\sigma(X) + \chi(X) = 0 \mod 4$. To do this we first observe that $c_1(X)$ is a *characteristic element* of the intersection form, which means that it has the property that

$$c_1(X) \cdot x = x^2 \mod 2 \text{ for all } x \in H^2(X; \mathbb{Z}).$$

This follows from the fact that $\mathrm{red}_2(c_1(X)) = w_2(X)$ and the Wu formula, which implies that $w_2(X) \cdot y = y^2$ for all $y \in H^2(X; \mathbb{Z}/2)$.

Now we need to use the fundamental fact from algebra that whenever c is a characteristic element of a unimodular symmetric form λ over $\mathbb{Z}$, then $c^2 = \sigma(\lambda) \mod 8$.

In our case we thus obtain

$$\begin{aligned} 3\sigma(X) + 2\chi(X) = c_1(X)^2 = \sigma(X) &\mod 8 \\ \Leftrightarrow 2(\sigma(X) + \chi(X)) = 0 &\mod 8 \\ \Leftrightarrow \sigma(X) + \chi(X) = 0 &\mod 4. \end{aligned}$$

To obtain the partial converse we need to argue harder. First we reinterpret the number $\sigma(X) + \chi(X)$ as follows.

We can write $\sigma(X) = b_2^+(X) - b_2^-(X)$ and $\chi(X) = 2 - 2b_1(X) + b_2^+(X) + b_2^-(X)$. It thus follows that

$$\sigma(X) + \chi(X) = 2(1 - b_1(X) + b_2^+(X)).$$

Now let us assume that $b_1(X) = 0$. Then the statement $\sigma(X) + \chi(X) = 0 \mod 4$ translates to $b_2^+(X)$ being odd.

Now we need to make a case by case study. Suppose first that the intersection form of X is odd. We may assume that it is of the form $n\langle 1\rangle \oplus m\langle -1\rangle$ and potentially m could be zero. This is because if the intersection form of a smooth 4-manifold is definite then it is standard by Donaldson, and it cannot be negative definite because $b_2^+(X) \neq 0$. We can write $n = 2k+1$ and call the basis elements of the form by $(e_1, \ldots, e_{2k+1}, f_1, \ldots, f_m)$. A characteristic element of this form is given by

$$d = \sum_{i=1}^{n} e_i + \sum_{j=1}^{m} f_j$$

We consider the element

$$c = 3e_1 + \sum_{i=1}^{k} 3e_{2i} + e_{2i+1} + \sum_{j=1}^{m} f_j$$

which is a characteristic element since $c-d$ is in the image of multiplication by 2. Thus $\mathrm{red}_2(c) = w_2$ as needed for part (1).

To see the second equation we calculate

$$c^2 = 9 + 9k + 1 - m = 9k + 10 - m$$

On the other hand we have

$$\begin{aligned} 3\sigma(X) + 2\chi(X) &= 3(2k+1-m) + 2(2+2k+1+m) \\ &= 6k + 3 - 3m + 4 + 4k + 2 + 2m \\ &= 10k + 9 - m \end{aligned}$$

hence c does the job.

Now suppose that the intersection form of X is even. Then it cannot be definite, again by Donaldsons result. Thus it is indefinite and even and hence determined by its signature and its rank. Since the intersection form is even we already know that the signature is divisible by 8 and hence the form is isomorphic to

$$kE_8 \oplus lH$$

where H stands for a hyperbolic form. Moreover since $b_2^+(X)$ is odd it follows that l must be odd.

Since the form is even, 0 is a characterisitc element. Let us denote the hyperbolic basis of the first summand of the H's by $\{a, b\}$. We calculate that

$$\begin{aligned} 3\sigma(X) + 2\chi(X) &= 3 \cdot 8k + 2(2 + 8k + 2l) \\ &= 24k + 16k + 4 + 4l \\ &= 40k + 4(l+1) \\ &= 8(5k + \tfrac{l+1}{2}) = 8q \end{aligned}$$

Notice that $q \in \mathbb{Z}$ since l is assumed to be odd.

Then we consider the element

$$c = 2(qa + b)$$

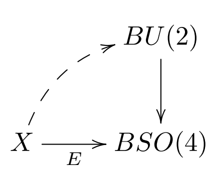
and calculate that

$$c^2 = 4(2q) = 8q = 3\sigma(X) + 2\chi(X)$$

as needed.

We end with the observation about the second obstruction:

Recall that for the lifting problem



there are two obstructions, the first one being

$$\beta(w_2(E)).$$

If this obstruction vanishes then we choose a lift $c \in H^2(X;\mathbb{Z})$ of w_2 and the previous arguments then show that the next obstruction is given by the element

$$c^2 - p_1(E) - 2e(E) \in H^4(X;\mathbb{Z})$$

In the case where $E = \tau_X$ the signature formula implies that $p_1(E) = 3\sigma(X)$ as claimed in the beginning.

The following remains open for the moment:

Question 1. If X is a general smooth oriented 4-manifold and $\sigma(X) + \chi(X)$ is divisible by 4, does X have an almost complex structure?

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