Comparing K- and L-theory in the Case of Complex C*-algebras

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K- and L-theory of C*-algebras

Let A be a complex (unital) C^*-algebra. We recall that this just means that A is a Banach-algebra over the complex numbers together with an involution $^*: A \to A$ that behaves particularly nice with the norm on A: it fulfills the socalled C^*-identity:

$$\|x^*x\| = \|x\|^2 \text{ for all } x \in A.$$

Associated to such a C^*-algebra A is a topological K-theory *spectrum* $\mathbb{KU}(A)$ such that

$$\pi_i(\mathbb{KU}(A)) \cong KU_i(A) \text{ for all } i \in \mathbb{Z}.$$

On the other hand we can forget that A was a C^*-algebra and just remember the involutive ring A. Associated to such rings are algebraic L-groups $L(A)$, which depend on a decoration like *projective* and *free*, written as $L^{\langle p \rangle}(A)$ resp. $L^{\langle h \rangle}(A)$, moreover for any decoration α there is a *symmetric* and a *quadratic* version, written as $L^*_{\langle \alpha \rangle}(A)$ and $L^{\langle \alpha \rangle}_*(A)$ respectively. There is a forgetful map from the free to the projective decoration. By construction these groups are 4-periodic, and for rings of the form $\mathbb{Z}\pi$, where π is the fundamental group of a Poincaré complex X the L-groups are the surgery obstruction groups that appear in the surgery exact sequence. Due to Ranicki, there is a purely algebraic description of the L-groups in terms of bordism classes of algebraic Poincaré chain complexes, and using this, Ranicki constructed L-theory *spectra*, which we will denote by $\mathbb{L}^{\langle \alpha \rangle}_*(A)$.
There is a lot one can say about the L-groups of complex C^*-algebras, partly summarized in the following

Lemma. *If A is a complex C^*-algebra, we have the following properties*

1. *The symmetrization map $L^{\langle \alpha \rangle}_*(A) \to L^*_{\langle \alpha \rangle}(A)$ is an isomorphism. This is because $2 \in A^\times$.*
2. *L-groups are 2-periodic, since $i \in A$.*
3. *The forgetful map $L^{\langle h \rangle}_0(A) \to L^{\langle p \rangle}_0(A)$ sits in the exact sequence*

$$0 \longrightarrow \widehat{H}^1(C_2; \widetilde{K}_0(A)) \longrightarrow L^{\langle h \rangle}_0(A) \longrightarrow L^{\langle p \rangle}_0(A) \longrightarrow \widehat{H}^2(C_2; \widetilde{K}_0(A)) \longrightarrow 0$$

4. *The forgetful map $L^{\langle h \rangle}_1(A) \to L^{\langle p \rangle}_1(A)$ is an isomorphism. This follows from parts (2) and (3) using the Rothenberg exact sequence.*

Surprisingly, topological K-theory and algebraic L-theory of C^*-algebras are not unrelated: One can prove the following

Proposition. *Let A be a complex C^*-algebra. Then there is a natural isomorphism*

$$L^0_{\langle p \rangle}(A) \longrightarrow KU_0(A)$$

Proof. An element of $L^0_{\langle p \rangle}(A)$ is represented by a symmetric non-degenerate form φ over some projective module P over A. Similarly as in the case of $A = \mathbb{C}$ one can use the functional calculus of C^*-algebras to decompose $P \cong P^+ \oplus P^-$ such that φ is diagonal with respect to this decomposition and is positive definite on P^+ and negative definite on P^-. Moreover up to unitary equivalence, this decomposition is unique. The kernel of this map are precisely the hyperbolic forms, moreover any projective module has a positive definite form over it, using some embedding in a free module and taking the standard form there. This implies that the map is bijective. □

Comparing the K- and L-theory groups

Using part (1) of the previous Lemma we get the following

Corollary. *For any complex C^*-algebra there is a natural isomorphism*

$$L^{\langle p \rangle}_0(A) \longrightarrow KU_0(A)\,.$$

More elaborate, but still true is the following

Proposition. *For a complex C^*-algebra, we also have a natural isomorphism*

$$L^{\langle h \rangle}_1(A) \longrightarrow KU_1(A)\,.$$

Proof. In [5] Rosenberg gives an argument that uses a topological version of L-theory. In [3] Miller constructs a zig-zag of isomorphisms, using a bordism type description of topological K-theory. In [1] the authors remark that their construction directly gives a map from L- to K-theory but we think that this uses Miller's results. In any case, it is much less obvious and direct than the case of the previous proposition. □

Remark. *Notice that even if there is a direct map as in the proposition, to compare $L^{\langle p \rangle}_1(A)$ to $KU_1(A)$ we still have a zig-zag*

$$L^{\langle p \rangle}_1(A) \xleftarrow{\cong} L^{\langle h \rangle}_1(A) \longrightarrow KU_1(A)\,.$$

Motivated by this we now have the following

Naive Goal. *Construct a (zig-zag of) natural isomorphism(s) of spectra valued functors*

$$\mathbb{L}^{\langle p \rangle}_* \xrightarrow{\ \tau\ } \mathbb{KU}$$

from quadratic L-theory with projective decoration to topological K-theory.

The situation is even interesting when viewing L and K-theory spectra in the stable homotopy category. There we have the following. Given any C^*-algebra A, the spectrum $\mathbb{KU}(A)$ is a module spectrum over KU, the usual complex topological K-theory spectrum.
Similarly, using pairings Ranicki constructs, one sees that $\mathbb{L}_*(A)$ is a module spectrum over $\mathbb{L}^*(\mathbb{C})$, see e.g. [4, Appendix B].
Suppose for the moment that in the homotopy category we have an isomorphism of homotopy ring spectra

$$\mathbb{L}^*(\mathbb{C}) \cong \mathrm{KU}$$

There is a result by Bousfield that characterizes the homotopy category of module spectra over KU, which implies that two modules M and N over KU with abstractly isomorphic homotopy groups (as modules over $\pi_*(\mathrm{KU})$) are weakly equivalent as module spectra (even that every isomorphism of homotopy groups lifts to a weak equivalence inducing this map on homotopy groups).
In particular the previous two propositions show that one would get an isomorphism in the homotopy category

$$\mathbb{L}_*(A) \cong \mathbb{KU}(A)$$

The Problem of Having to Invert 2

The problem with the above argument is the following.

$\mathbb{L}^*(\mathbb{C})$ is not isomorphic to KU

This is due to the following observations. We can look at the spectra after localizing at the prime 2.
Now it is well-known that $\mathrm{KU}_{(2)}$ does not split as a wedge of Eilenberg-MacLane spectra, which for example follows from the computation

$$\widetilde{\mathrm{KU}}(\mathbb{R}P^4) \cong \mathbb{Z}/4\mathbb{Z},$$

but we have the following

Proposition ([6, Thm A])**.** *The 2-localized spectrum $\mathbb{L}^*(\mathbb{C})_{(2)}$ splits as a wedge of Eilenberg-MacLane spectra.*

Proof. From [4, Appendix B] it follows that $\mathbb{L}^*(\mathbb{C})$ is an $\mathbb{L}^*(\mathbb{Z})$-algebra. Moreover there is the Sullivan-Ranicki orientation, which is a ring map

$$\mathrm{MSTop} \longrightarrow \mathbb{L}^*(\mathbb{Z})\,.$$

Composing this with the obvious ring map $\mathrm{MSO} \to \mathrm{MSTop}$ this shows that $\mathbb{L}^*(\mathbb{C})$ is an algebra over MSO.
Now in [6, Thm A] it is shown that any module E over MSO will split 2-locally. □

$\mathbb{L}^*(\mathbb{C})[\frac{1}{2}]$ is indeed isomorphic to $\mathrm{KU}[\frac{1}{2}]$

Corollary. *For any complex C^*-algebra there is an isomorphism in the stable category $\mathbb{L}_*(A)[\frac{1}{2}] \cong \mathbb{KU}(A)[\frac{1}{2}]$.*

Remark. *Again from results of Bousfield we can surely deduce that $\mathbb{L}^*(\mathbb{C})[\frac{1}{2}]$ and $\mathrm{KU}[\frac{1}{2}]$ are equivalent as module spectra over $\mathrm{KO}[\frac{1}{2}]$. This is because as mentioned, $\mathbb{L}^*(\mathbb{C})[\frac{1}{2}]$ is a module over $\mathbb{L}^*(\mathbb{R})[\frac{1}{2}]$ and obviously $\mathrm{KU}[\frac{1}{2}]$ is a module over $\mathrm{KO}[\frac{1}{2}]$. Sullivan proved that there is an equivalence $\mathbb{L}_*(\mathbb{Z})[\frac{1}{2}] \simeq \mathrm{KO}[\frac{1}{2}]$ and from [5] it then follows that $\mathrm{KO}[\frac{1}{2}]$ is equivalent to $\mathbb{L}^*(\mathbb{R})[\frac{1}{2}]$ as ring spectra, because it suffices to see that the equivalence respects the units in the homotopy groups that induce the periodicity.*

We hence see that we need to adjust our goal and insert the construction of inverting 2 on the level of spectra:

Goal. *Construct a (zig-zag of) natural isomorphism(s) of spectra valued functors*

$$\mathbb{L}_*[\tfrac{1}{2}] \xrightarrow{\ \tau\ } \mathbb{KU}[\tfrac{1}{2}]$$

We want to note that the abstract isomorphism as in the last Corollary is not sufficient to deduce that we actually get a *natural* isomorphism in the stable category, needless to say that we are able to lift this to a category of spectra. The reason that we want to have a natural transformation between the functors to an honest category of spectra is that on the level of a model of the stable category we are able to do constructions like colimits and limits, which is not possible in general in the stable category.

Applications

For example we could nicely compare the L-theoretic Farrell-Jones conjecture to the Baum-Connes conjecture via the following commutative diagram

$$\begin{array}{ccc}
\mathcal{H}^G_*\big(\underline{E}G; \mathbb{L}_*(\mathbb{Z})[\frac{1}{2}]\big) & \xrightarrow{\ \mathrm{FJ}\ } & L_*(\mathbb{Z}G)[\frac{1}{2}] \\
\cong\downarrow & & \downarrow \\
\mathcal{H}^G_*\big(\underline{E}G; \mathbb{L}_*(\mathbb{Q})[\frac{1}{2}]\big) & \longrightarrow & L_*(\mathbb{Q}G)[\frac{1}{2}] \\
\cong\downarrow & & \downarrow \\
\mathcal{H}^G_*\big(\underline{E}G; \mathbb{L}_*(\mathbb{R})[\frac{1}{2}]\big) & \longrightarrow & L_*(\mathbb{R}G)[\frac{1}{2}] \\
(\dagger)\downarrow & & \downarrow \\
\mathcal{H}^G_*\big(\underline{E}G; \mathbb{L}_*(\mathbb{C})[\frac{1}{2}]\big) & \longrightarrow & L_*(\mathbb{C}G)[\frac{1}{2}] \\
\cong\downarrow & & \downarrow \\
\mathcal{H}^G_*\big(\underline{E}G; \mathbb{L}_*(C^*_r(-))[\frac{1}{2}]\big) & \longrightarrow & L_*(C^*_rG)[\frac{1}{2}] \\
\cong\downarrow & & \downarrow \\
\mathcal{H}^G_*\big(\underline{E}G; \mathbb{KU}[\frac{1}{2}]\big) & \xrightarrow[\mathrm{BC}]{} & KU_*(C^*_rG)[\frac{1}{2}]
\end{array}$$

In particular it would follow that if the BC-assembly map is rationally injective, then so is the L-theoretic Farrell-Jones assembly map, which in turn implies the Novikov conjecture for the group G.
We would like to remark that one way of proving the injectivity of the map (†) is by introducing the real versions of group C^*-algebras and then proving that L-theory of real C^*-algebras is isomorphic to real topological K-theory after inverting 2. Then on the level of K-theory one knows that complexification is split injective after inverting 2. It might be interesting to find an intrinsic reason that for a finite group H, the complexification map

$$L_*(\mathbb{R}H)[\tfrac{1}{2}] \longrightarrow L_*(\mathbb{C}H)[\tfrac{1}{2}]$$

is (split) injective.

References

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